A novel iterative solution of the three dimensional electric field integral equation

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Abstract — **A novel forward backward iterative scheme for solving the 3D electric field integral equation is presented. This method postulates the use of a local buffer region to suppress spurious edge effects that would otherwise grow to dominate the result. Results are presented illustrating the convergence of the algorithm.**

1 INTRODUCTION

The Electric Field Integral Equation (EFIE) offers a full wave formulation to the problem of electromagnetic wave scattering from a perfectly conducting object. However numerical solution of the EFIE results in a dense linear system which, for large problems, is impossible to store let alone invert. Instead iterative solutions are used which do not require the explicit inversion of the matrix but rather sequentially build the solution. Recently there has been much research into what can be termed physically inspired iterative solutions, or informally 'current marching' methods. Examples include the Method of Ordered Interactions of [1] and the forward/backward method of [2], both of which were applied to two-dimensional problems. A three-dimensional version was presented in [3] for application to a Magnetic Field Integral Equation (MFIE) formulation of a scattering problem. However the MFIE is only applicable for scattering from closed bodies and one must use the EFIE if one wishes to consider scattering from an open body. This paper presents a current marching algorithm which is applicable to the three dimensional EFIE.

2 ELECTRIC FIELD INTEGRAL EQUATION

We consider a perfectly conducting scatterer is illuminated by a source which induces currents on its surface S. We define the incident electric field $\mathbf{E}^i$ as the field that would exist in the absence of the scatterer. The total field anywhere in space is then given by the sum of the incident and scattered electric fields where the scattered field, $\mathbf{E}^s$ is given

by the following integral expression

$$\mathbf{E}^s = -\jmath\omega\mathbf{A} - \nabla\phi \qquad (1)$$

where $\mathbf{A}$ and ϕ are the magnetic vector potential and scalar potential respectively. Applying the requirement of zero tangential fields at a point on the surface of the scatterer results in the EFIE which can be converted to a matrix equation by the introduction of basis functions and a testing procedure [4]. The resultant equation is

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \qquad (2)$$

where $\mathbf{Z}$ is a dense complex valued $N \times N$ matrix, where N is the number of basis functions used to discretise the surface current. $\mathbf{V}$ is a vector of size N containing information about the incident field on the scatterer while the vector $\mathbf{I}$ represents the (unknown) amplitudes of the N basis functions.

3 FORWARD/BACKWARD ALGORITHM FOR THREE DIMENSIONAL SCATTERERS

The necessity to sample the current at a high rate (typically around ten basis functions per wavelength) means that for problems of practical size it becomes impossible to store, let alone invert the impedance matrix $\mathbf{Z}$. Instead the matrix equation is typically solved using an iterative procedure such as the method of conjugate gradients. Recently there has been much interest in the concept of physically inspired iterative solvers. These solve for the unknown current amplitudes $\mathbf{I}$ in a manner that attempts to mimic the physical processes that create the current. Specifically this involves decomposing the scatterer into sub-regions and 'marching' a solution for the current along the scatterer surface from sub-region to sub-region. The solution at processed sub-regions is used to set up the problem to be solved at the next sub-region and so on. Mathematically the process involves decomposing the $\mathbf{Z}$ matrix into blocks, the $\tilde{\mathbf{Z}}_{ij}$ block containing the interactions between the basis functions residing in the ith and j^{th} sub-regions on the scatterer. Each iteration of a forward/backward algorithm involves solving two equations. The first equation is solved for $i = 1 \ldots N$ in turn and is termed the forward

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sweep.

$$\tilde{\mathbf{Z}}_{ii}\tilde{\mathbf{I}}_i^{(k)} = \tilde{\mathbf{V}}_i - \sum_{j=1}^{i-1}\tilde{\mathbf{Z}}_{ij}\tilde{\mathbf{I}}_j^{(k)} - \sum_{j=i+1}^{N}\tilde{\mathbf{Z}}_{ij}\tilde{\mathbf{I}}_j^{(k-1)} \quad (3)$$

where $\tilde{\mathbf{V}}_i$ and $\tilde{\mathbf{I}}_i$ are the appropriate sub-vectors of $\mathbf{V}$ and $\mathbf{I}$ respectively. Equation (3) is a matrix equation for the currents on sub-region i where the right hand side incident fields have been modified by including the effects of the currents already computed on the other sub-regions. As it involves a matrix of relatively low order it can be efficiently solved using a conjugate gradient solver. The second equation is solved for $i = N \dots 1$ in turn and corresponds to a backward sweep.

$$\tilde{\mathbf{Z}}_{ii}\tilde{\mathbf{I}}_i^{(k+1)} = \tilde{\mathbf{V}}_i - \sum_{j=1}^{i-1}\tilde{\mathbf{Z}}_{ij}\tilde{\mathbf{I}}_j^{(k)} - \sum_{j=i+1}^{N}\tilde{\mathbf{Z}}_{ij}\tilde{\mathbf{I}}_j^{(k+1)} \quad (4)$$

Again this can be efficiently solved using a conjugate gradient solver. We deem an iteration to be one complete forward sweep followed by a complete backward sweep.

We can examine the convergence or otherwise of the iterative process thus defined by computing how well the governing matrix equation (2) is satisfied at each iteration. We define the boundary condition percentage error associated with the ith basis function after completion of the nth iteration by

$$e_i^{(n)} = 100.0 \left| \frac{V_i - \sum_{i=1}^{N} Z_{ij} I_j^{(2n)}}{V_i} \right| \quad (5)$$

The convergence or otherwise of the method can be monitored by computing the average value of this error $\bar{e}^{(n)}$ after each iteration.

Whilst appealing in its intuitive underpinning the iterative process described above fails when applied to a relatively easy three dimensional problem. Indeed we applied it to a simple example involving scattering from a metallic square plate of side 2λ centred at (0,0,0) being illuminated by a half-wave wavelength z oriented dipole located at $(0, -10\sqrt{3}, 10)$ and radiating at $300MHz$. For this example the sub-regions consisted of 6 rectangular groupings of basis functions running along the whole width of the plate as shown in figure (1). Table (1) contains the average boundary condition error at each iteration step and illustrates how the iterative method as applied above quickly diverges To understand the failure of the iterative process as described above we must appreciate that each subregion is effectively considered in isolation when computing the currents on it on any given sweep. Consider the first step in the first forward sweep.

Iteration n	Average error
1	282.7
2	13270.8
3	758249.4
4	43163263.5

Table 1: Average error at the boundary condition as a function of iteration number for naive forward/backward approach.

The current estimate $\mathbf{I}^{(1)}$ is initialised to zero and (3) reduces to

$$\tilde{\mathbf{Z}}_{11}\tilde{\mathbf{I}}_1^{(1)} = \tilde{\mathbf{V}}_1 \quad (6)$$

Essentially we are treating sub-region one as a physically isolated scatterer. As a consequence the computed current $\tilde{\mathbf{I}}_1^{(1)}$ will display the singular behaviour which characterise the current at the edge of scatterers. While this edge effect is correct and desirable for any edges of the subregion which coincide with the actual edges of the entire scatterer any edge effect at the boundary with subregion two is undesirable. Were the scatterer problem solved as a whole no such effect would appear and so its appearance is spurious and should be suppressed. However the naive iterative method as suggested by equations (3,4) fail to do this. Consider the next problem to be solved in the initial forward sweep. We have

$$\tilde{\mathbf{Z}}_{22}\tilde{\mathbf{I}}_2^{(1)} = \tilde{\mathbf{V}}_2 - \tilde{\mathbf{Z}}_{21}\tilde{\mathbf{I}}_1^{(1)} \quad (7)$$

Here the inaccuracy due to the spurious edge effect is allowed propagate and distort the computation of $\tilde{\mathbf{I}}_2^{(1)}$. In addition $\tilde{\mathbf{I}}_2^{(1)}$ will also manifest spurious singularity effects at any edge which does not correspond to a physical edge of the scatterer.

However we can circumvent this problem quite simply by introducing a certain amount of redundancy into our computations. These extra calculations will be shown to dramatically improve the stability of the iterative process at the cost of a slightly higher computational burden. We identify for each sub-region 'buffer regions' (see figure (1)) which are those areas of the scatterer immediately adjacent to the boundary of the sub-region in the direction that we are marching the solution. Note that the definition of the buffer region thus depends on whether we are on the forward or backward sweep of the iterative process. The idea is to include the interactions with the basis functions in this buffer region to suppress the unphysical current singularities which, as we saw, are introduced by the abrupt termination of each sub-region. Mathematically we replace the previous forward and backward sweeps

with revised versions

$$\tilde{\mathbf{Y}}_{ii}\tilde{\mathbf{J}}_i^{(k)} = \tilde{\mathbf{W}}_i - \sum_{j=1}^{i-1} \tilde{\mathbf{Y}}_{ij}\tilde{\mathbf{I}}_j^{(k)} - \sum_{j>i,j\notin b(i)} \tilde{\mathbf{Y}}_{ij}\tilde{\mathbf{I}}_j^{(k-1)}$$
(8)

$$\tilde{\mathbf{Y}}_{ii}\tilde{\mathbf{J}}_i^{(k+1)} = \tilde{\mathbf{W}}_i - \sum_{j<i,j\notin b(i)} \tilde{\mathbf{Y}}_{ij}\tilde{\mathbf{I}}_j^{(k)} - \sum_{j=i+1}^{N} \tilde{\mathbf{Y}}_{ij}\tilde{\mathbf{I}}_j^{(k+1)}$$
(9)

where $b(i)$ represents the appropriate buffer region (depending on whether we are marching forward or backward). $\tilde{\mathbf{Y}}_{ii}$ supplements $\tilde{\mathbf{Z}}_{ii}$ with information about the interaction between basis functions in i and those in the appropriate (forward or backward) buffer region.

$$\tilde{\mathbf{Y}}_{ii} = \begin{bmatrix} \tilde{\mathbf{Z}}_{ii} & \tilde{\mathbf{Z}}_{ib(i)} \\ \tilde{\mathbf{Z}}_{b(i)i} & \tilde{\mathbf{Z}}_{b(i)b(i)} \end{bmatrix}$$
(10)

while $\tilde{\mathbf{Y}}_{ij}$ supplements $\tilde{\mathbf{Z}}_{ij}$ with information about the interaction between basis functions in j and those in the appropriate buffer region.

$$\tilde{\mathbf{Y}}_{ij} = \begin{bmatrix} \tilde{\mathbf{Z}}_{ij} \\ \tilde{\mathbf{Z}}_{b(i)j} \end{bmatrix}$$
(11)

$\tilde{\mathbf{J}}_i^{(k)}$ and $\tilde{\mathbf{W}}_i$ are given by

$$\tilde{\mathbf{J}}_i^{(k)} = \begin{bmatrix} \tilde{\mathbf{I}}_i^{(k)} \\ \tilde{\mathbf{I}}_{b(i)} \end{bmatrix}$$
(12)

$$\tilde{\mathbf{W}}_i = \begin{bmatrix} \tilde{\mathbf{V}}_i \\ \tilde{\mathbf{V}}_{b(i)} \end{bmatrix}$$
(13)

This modification ensures that local information from basis functions in the buffer region suppresses any spurious edge effect that would otherwise be present. Computationally it is a little more cumbersome to solve equations (8,9) due to the fact that the matrices are of higher order. The currents calculated in the buffer region $\tilde{\mathbf{I}}_{b(i)}$ are redundant in the sense that they are only computed to keep the currents in sub-region i under control. They are overwritten when we move to the next sub-region. However in practice the buffer region is quite small and the increased computational overhead is offset by the rapid convergence of the algorithm.

3.1 Recursive algorithm

Initial implementations of our algorithm proceeded by breaking a scatterer into rectangular groups of basis functions which spanned the whole width of the scatterer as illustrated in figure (1). However, with this approach, as the scatterer size increases the size of each subregion grows accordingly. When
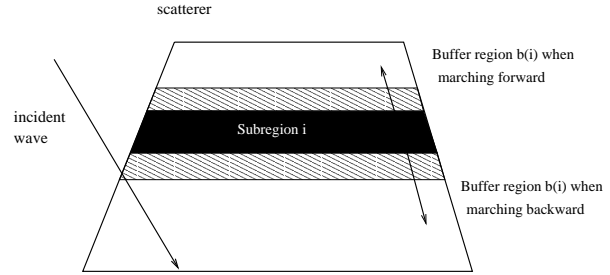

Figure 1: Subregion along with 'buffer regions' to ensure stability of current marching method.

one adds in the extra computational burden imposed by the buffer region interactions the solution of the matrix equations (8,9) quickly becomes quite onerous. However it is possible to recursively apply the ideas presented in the previous section to the efficient solution of these equations also. As depicted in figure (2) each sub-region can in turn be further subdivided into smaller sub-regions and the matrix equations (8,9) are solved by marching currents back and forth within sub-region i and its buffer. Essentially the scattering problem is solved by a process of forward and backward sweeps, where the local problems within each forward/backward sweep are solved by a process of sweeping left and right. Again we must be careful to define suitable buffer regions to suppress unwanted edge effects.
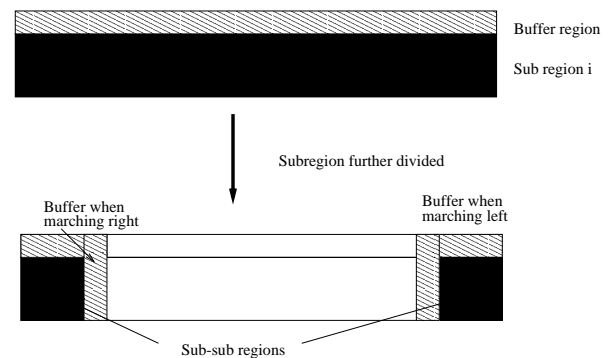

Figure 2: Subregion i can be further subdivided facilitating efficient current marching solution of equations (8) and (9).

4 RESULTS

We have applied the techniques described in this paper to a number of problems. First we consider the problem of scattering by metallic square plate of side 15λ centred at the point $(0,0,0)$ containing an aperture of side 1λ centred at $(-2,-2,0)$. The plate is illuminated by a dipole located at

$(0, -10\sqrt{3}, 10)$ radiating at 900MHz. For the purpose of applying the algorithm the plate was subdivided into 10 rectangular subregions each of which spanned the width of the plate. The currents were then marched forward and backward along the plate surface. Currents within each subregion were obtained by marching left/right as per section (3.1). At this level each sub-sub-region (see figure(2)) was a square section of plate of side 1.5λ. The buffer zone was such that the entire computational region for each problem was a square of size 2λ. Table (2) shows the average boundary condition error versus the iteration number.

Iteration n	Average error
1	13.6146
2	2.27561
3	0.42557

Table 2: Average error at the boundary condition as a function of iteration number for example one.

The second example involves a right angled finite wedge composed of two perfectly conducting plates of side 2λ meeting along a common edge. The first plate is centred at $(0, 0, 0)$ while the second is centred at $(0, 1, 1)$ Again the source is a half-wave dipole located at $(0, -10\sqrt{3}, 10)$, this time radiating at 300MHz. Each plate was subdivided into 6 rectangular subregions each of which spanned the width of the plate. The currents were then marched forward and backward along the wedge structure. As in the previous example currents within each subregion were obtained by marching left/right as per section (3.1). Again each such computational region was a square section of plate of side 2λ. The scattered fields along a straight line running from $(1.5, -20, 10)$ to $(1.5, 20, 10)$ were calculated after each iteration. We note the rapid convergence of the scattered fields, the solution after 3 iterations being essentially identical to a reference solution obtained using a conjugate gradient method. The algorithm was also applied to a larger finite right angled wedge with sides equal to 5λ. Again rapid convergence was observed with the average boundary error falling to 6% after only three iterations.

5   CONCLUSIONS

We have presented a modified forward/backward algorithm for iteratively solving the three-dimensional EFIE. Stability was achieved by the adoption of buffer regions which suppress spurious edge effects which would otherwise grow to dominate the solution. Future work will see
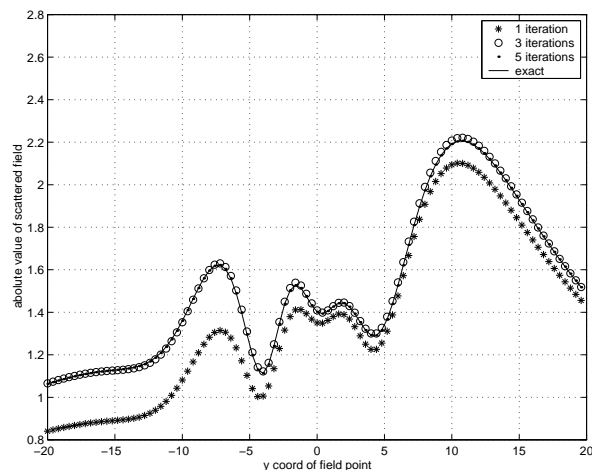

Figure 3: Convergence of scattered fields over wedge.

the use of acceleration schemes such as the Fast Far-Field Algorithm [5] to expedite the scattered field computations that represent the bottleneck in the solution.

Acknowledgments

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