Numerical Evaluation of the Singularity Arising in the Two-Dimensional Volume Electric Field Integral Equation

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***Abstract*—A numerical procedure for the treatment of the singularity arising in the Method of Moments (MoM) solution of the two-dimensional volume Electric Field Integral Equation (EFIE) is introduced in this paper. The procedure expresses the singular integral in terms of an analytic function and employs a singularity isolation process coupled with numerical quadrature along the domain perimeter to evaluate the self-terms. Numerical results are presented comparing the method to conventional techniques. In particular, fields scattered from a dielectric cylinder, discretised with pulse basis and delta testing functions, are computed and compared against those from a reference Mie series solution. The results obtained using the numerical procedure described are shown to be superior.**

I. Introduction

Accurate numerical evaluation of the singular integrals that arise in the Method of Moments (MoM) solution of the EFIE is of fundamental importance to the accuracy of electromagnetic wave scattering solvers. The singularity occurs in the Green's function of the self-term elements of the MoM impedence matrix, where the testing and source domains coincide. Due to the large contribution of the self-term component it is important to evaluate its effect accurately [1].

There has been extensive research carried out in numerical and analytical evaluation of singular integrals [2]–[4]. In this paper we outline a novel singularity isolation approach for the numerical evaluation of the two-dimensional Green's function singularity. This technique is based on the seminal work by [5], [6] for the treatment of the self-term integral involving the three-dimensional Green's function, but differs in important details as we consider the two-dimensional Green's function based on the Hankel function. In this approach the self-term integral is split into two parts, namely an analytically-evaluated integral that isolates the singularity as well as a numerically-evaluated component.

We consider a two-dimensional dielectric cylinder characterised by a permittivity $\epsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$ for a TM^z configuration [1]. The object is illuminated by a plane wave with time variation $\exp(\jmath\omega t)$ (which is assumed and suppressed in what follows). The corresponding integral equation can be expressed in terms of the unknown electric field $E_z(\mathbf{r})$ [1], [7]

$$E_z^i(\mathbf{r}) = E_z(\mathbf{r}) + \int_v E_z(\mathbf{r}')\, g(\mathbf{r},\mathbf{r}')\, O(\mathbf{r}')\, dv' \qquad (1)$$

where the two-dimensional Green's function is given by

$$g(\mathbf{r},\mathbf{r}') = \frac{\jmath}{4} H_0^{(2)}(k_b|\mathbf{r}-\mathbf{r}'|) \qquad (2)$$

where the background wavenumber is denoted by k_b and $H_0^{(2)}$ is the zero order Hankel function of the second kind. The electrical properties of the scatterer are described by the object function

$$O(\mathbf{r}') = k^2(\mathbf{r}') - k_b^2. \qquad (3)$$

where $k(\mathbf{r}')$ is the wave number at a point in the scatterer. The object function is thus zero outside the scattering structure. Using the Method of Moments with m basis and testing functions [1], [7], Equation 1 is discretised resulting in

$$(\mathbf{I}+\mathbf{Z})\,\mathbf{x} = \mathbf{b} \qquad (4)$$

where $\mathbf{b}$ is a vector containing information about the incident field, $\mathbf{Z}$ is an $m \times m$ matrix containing coupling information between the basis functions and $\mathbf{x}$ contains the unknown basis function coefficients. In what follows we assume the use of pulse basis functions and Dirac-Delta testing functions.

II. Methodology

The numerical difficulties arise in evaluating the integral associated with the self-term diagonal matrix entries in Equation 4. A similar approach to [6] is undertaken, assuming the basis domain to be a triangular cell T[1]. Figure 1 illustrates a typical triangular domain T whose perimeter is made up

[1] In this work we assume triangular basis domains in order to better tessellate the scatterer and maximise accuracy when comparing to reference solutions. Square domains associated with FFT-expedited solutions of the volume EFIE can be easily handled, by considering each square as the union of two triangles

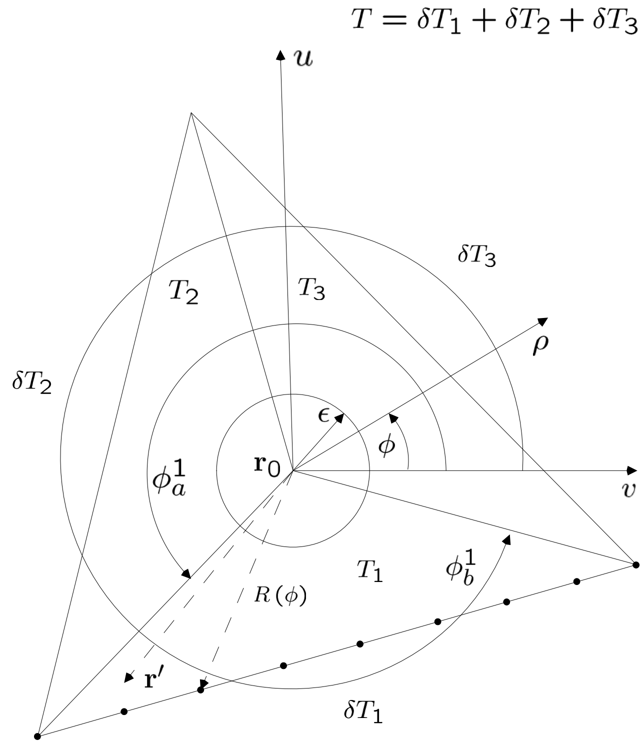


Fig. 1. Triangular cell T subdivided divided into subtriangles δT_i [6].

of three edges δT_i. $\mathbf{r}_0$ and $\mathbf{r}'$ are the observation point and integration points respectively, defined on a coordinate system (u, v). The auxiliary polar coordinate system is defined by (ρ, ϕ) where ρ is the distance between the integration and observation points. ϕ_a^i and ϕ_b^i are the angles associated with the endpoints of δT_i relative to the coordinate system (u, v). $R(\phi)$ is the distance between a point on δT_i and the observation point. It is thus a function of ϕ, where $\phi_a^i < \phi < \phi_b^i$.

We begin the procedure by substituting the singular Hankel function in Equation 2 with a power series expansion which is valid for small arguments x [1], [8]

$$H_0^{(2)}(x) \approx \left(1 - \frac{2\jmath}{\pi} \ln\left(\frac{\gamma x}{2}\right)\right) \tag{5}$$

where $\gamma = 1.781$. The integral to be evaluated is

$$\vartheta = \int_v H_0^{(2)}(|\mathbf{r} - \mathbf{r}'|)\, dv' \tag{6}$$

We note the singular behaviour of the integrand at $\mathbf{r}' = \mathbf{r}$. In order to evaluate ϑ the singularity is isolated inside a disc of radius ϵ and its contribution evaluated analytically. The contribution from the remainder of the triangle is numerically evaluated by dividing the triangle into three sub-triangles δT_i and summing the contribution from each sub-triangle, yielding

$$\vartheta = \vartheta_1 + \sum_{i=1}^{3} \vartheta_2^i \tag{7}$$

where

$$\vartheta_1 = \int_0^{2\pi} \lim_{\epsilon \to 0} \int_0^{\epsilon} \left(1 - j\frac{2}{\pi} \ln\left(\frac{\gamma k_b \rho}{2}\right)\right) \rho\, d\rho\, d\phi = 0 \tag{8}$$

$$\begin{aligned} \vartheta_2^i &= \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \int_{\epsilon}^{R(\phi)} \left(1 - \frac{2\jmath}{\pi} \ln\left(\frac{\gamma k_b \rho}{2}\right)\right) \rho\, d\rho\, d\phi \\ &= \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \left.\frac{\rho^2}{2}\right|_{\epsilon}^{R(\phi)} d\phi \\ &\quad - \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \int_{\epsilon}^{R(\phi)} \frac{2\jmath}{\pi} \ln\left(\frac{\gamma k_b \rho}{2}\right) \rho\, d\rho\, d\phi \end{aligned} \tag{9}$$

The singularity has thus been isolated inside a disc of radius ϵ whose integral, ϑ_1, has been analytically evaluated to zero in the limit. The problem is now reduced to numerically evaluating the triplet of integrals ϑ_2^i. Making the substitution

$$x = \frac{\gamma k_b \rho}{2} \tag{10}$$

yields [9]

$$\begin{aligned} \vartheta_2^i = &\int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \left.\frac{\rho^2}{2}\right|_{\epsilon}^{R(\phi)} d\phi \\ &- \frac{2\jmath}{\pi} \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \int_{x=\frac{\gamma k_b \epsilon}{2}}^{\frac{\gamma k_b R(\phi)}{2}} \ln(x) \left(\frac{2}{k_b \gamma}\right)^2 x\, dx\, d\phi. \end{aligned} \tag{11}$$

Then using the identity [8]

$$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \tag{12}$$

and the fact that

$$\lim_{x \to 0} \frac{x^2}{2} \ln x = 0 \tag{13}$$

we produce

$$
\begin{aligned}
\vartheta_2^i = & \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \frac{R(\phi)^2 - \epsilon^2}{2} d\phi \\
& - \frac{2\jmath}{\pi} \left(\frac{2}{k_b \gamma} \right)^2 \int_{\phi_A^i}^{\phi_B^i} \lim_{\epsilon \to 0} \left[\left(\frac{1}{2} \left(\frac{\gamma k_b R(\phi)}{2} \right)^2 \right. \right. \\
& \left. \ln \left(\frac{\gamma k R(\phi)}{2} \right) - \frac{1}{4} \left(\frac{\gamma k_b R(\phi)}{2} \right)^2 \right) \\
& \left. - \left(\frac{1}{2} \left(\frac{\gamma k_b \epsilon}{2} \right)^2 \ln \left(\frac{\gamma k_b \epsilon}{2} \right) - \frac{1}{4} \left(\frac{\gamma k_b \epsilon}{2} \right)^2 \right) \right] d\phi \\
= & \int_{\phi_A^i}^{\phi_B^i} \frac{R(\phi)^2}{2} d\phi \\
& - \frac{2\jmath}{\pi} \int_{\phi_A^i}^{\phi_B^i} \left(\frac{R(\phi)^2}{2} \ln \left(\frac{\gamma k R(\phi)}{2} \right) - \frac{R(\phi)^2}{4} \right) d\phi \\
= & \int_{\phi_A^i}^{\phi_B^i} \frac{R(\phi)^2}{2} - \frac{\jmath R(\phi)^2}{\pi} \ln \left(\frac{\gamma k R(\phi)}{2} \right) + \frac{\jmath R(\phi)^2}{2\pi} d\phi \\
= & \int_{\phi_A^i}^{\phi_B^i} R(\phi)^2 \left[\frac{1}{2} + \frac{\jmath}{2\pi} \left(1 - 2 \ln \left(\frac{\gamma k R(\phi)}{2} \right) \right) \right] d\phi.
\end{aligned} \tag{14}
$$

The final step is the evaluation of the three integrals ϑ_2^i over the intervals $\phi_a^i < \phi < \phi_b^i$. They can be numerically integrated by employing a Gaussian quadrature formula [1], [6] yielding finally

$$
\vartheta = \sum_{i=1}^{3} \vartheta_2^i \tag{15}
$$

$$
\simeq \sum_{i=1}^{3} \left(\sum_{j=1}^{\alpha} w_j f_i(\phi_j) \right) \tag{16}
$$

where $\{w_j\}_{j=1,\ldots,J}$ and $\{\phi_j\}_{j=1,\ldots,J}$ are the weights and abscissas adopted for each integral ϑ_2^i.

III. Results

The accuracy of the singularity isolation method outlined in this paper is investigated by comparing it against two classical approaches. These approaches are the circular-cell approximation [1], [10], where the singularity is evaluated analytically by approximating the basis domain shape by a circle of the same area, and the Hankel power-series expansion [1], [8] for small arguments. The analytical Mie series [11] is used to independently validate the accuracy of these methods. Numerical experiments are performed for scattering problems involving a cylinder illuminated by a TM^z plane wave radiating at a frequency of $f = 300$ MHz and propagating in the x-direction.

A. Example 1

We initially consider a cylinder centred at the origin with radius $r = 0.16\lambda$ and relative permittivity $\epsilon_r = 6$ embedded

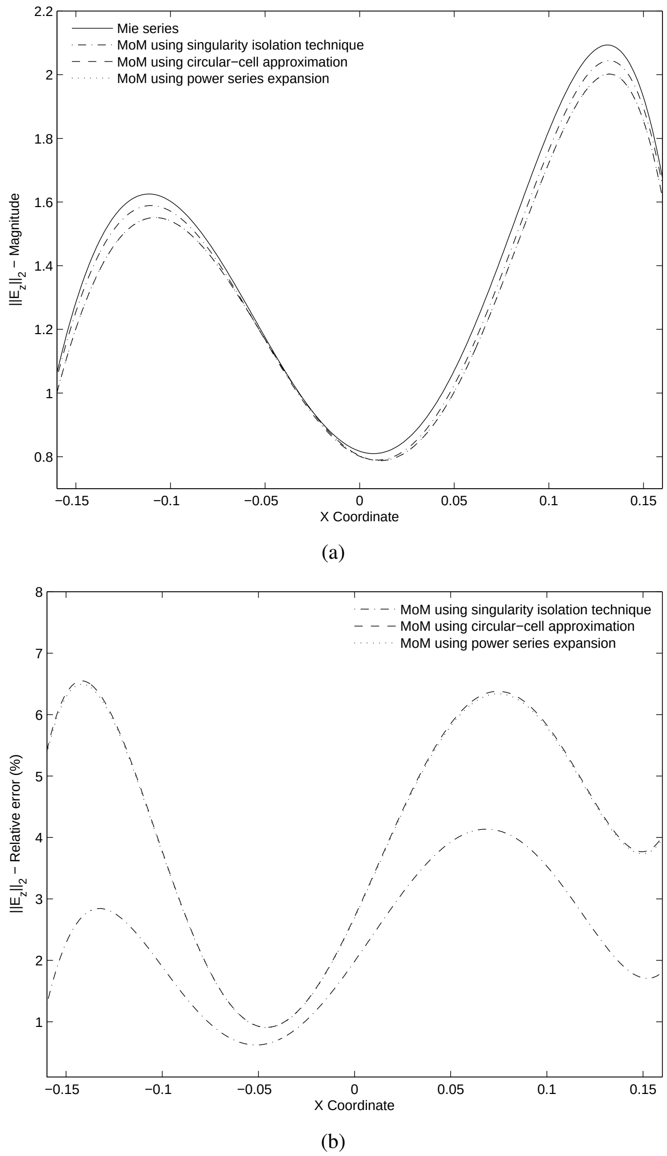
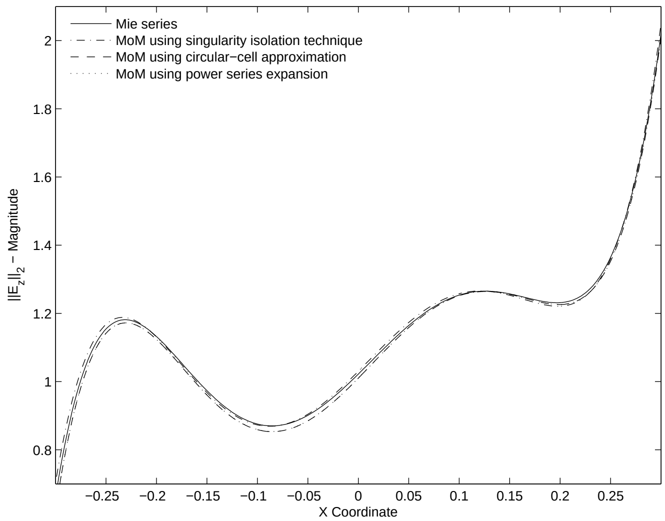
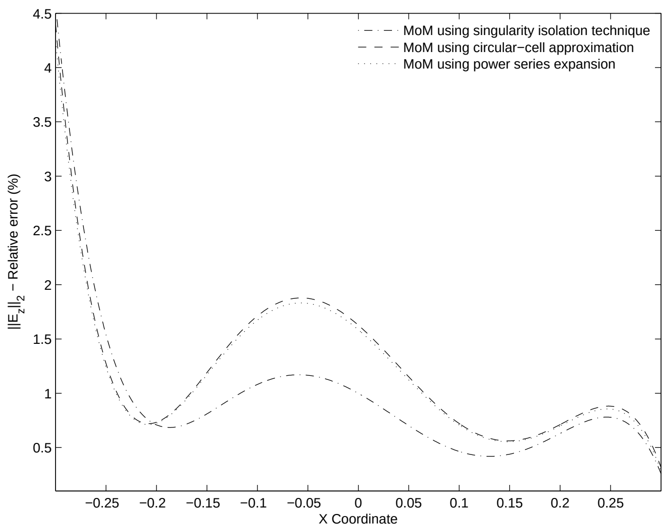


Fig. 2. Comparison of the numerical and Mie series results for the TM^z electric field within a dielectric cylinder with radius $r = 0.16\lambda$, relative permittivity $\epsilon_r = 6$ (Note: MoM using circular-cell approximation and MoM using power series expansion are overlapped)

in free space. The cylinder is discretised using $m = 416$ triangular cells and E^z is computed along a x-axis cut through the centre of the cylinder using 0.001λ increments. Computed E^z using the method of moments with various approaches to dealing with the singularity as well a Mie series reference solution are shown in Figure 2(a). Percentage relative errors are shown in Figure 2(b). The singularity isolation technique outlined in this paper achieves an average relative percentage (ARP) error of 2.34% while yielding a maximum absolute (MA) error of $0.061V/m$. In contrast the circular-cell approximation and the power series expansion technique achieve ARP errors of 4.11%, 4.08% and MA errors of $0.0986V/m$ and $0.0981V/m$ respectively.



(a)



(b)

Fig. 3. Comparison of the numerical and Mie series results for the TM^z electric field within a dielectric cylinder with radius $r = 0.3\lambda$, relative permittivity $\epsilon_r = 5$

B. Example 2

We now consider a cylinder of radius $r = 0.3\lambda$ and relative permittivity $\epsilon_r = 5$. The number of basis functions used is $m = 1624$ and E^z is again computed along a cut along the x-axis through the cylinder centre. The singularity isolation technique again outperforms the classical approaches with an ARP error of 0.954% and MA error of $0.0097V/m$. This is highlighted in Figures 3(a) and 3(b) demonstrating the improved accuracy of the singularity isolation technique presented in this paper. The associated ARP and MA errors for the circular-cell approximation and the power series expansion technique are 1.21%, 1.177% and $0.0262V/m$ and $0.0256V/m$. respectively.

IV. CONCLUSION

A technique for the numerical treatment of the singularity associated with the two-dimensional Green's function for a volume EFIE formulation is presented in the paper. The outlined method divides the singular integrand into an analytical part, which isolates the singularity, and a triplet of integrals which are numerically evaluated along the triangle perimeter. Numerical results are presented which compare the electric field calculated using the technique outlined in this paper with those of conventional approaches and a Mie series analytic reference solution. The numerical results demonstrate the improved accuracy of the scattered field solution when using the proposed method compared to conventional techniques.

REFERENCES

[1] A. F. Peterson, S. L. Ray, and R. Mittra, *Computational Methods for Electromagnetics*, 1st ed., ser. IEEE Press Series on Electromagnetic Wave Theory. Piscataway, New Jersey, USA: Wiley-IEEE Press, 1997.

[2] Y.-F. Jing, T.-Z. Huang, Y. Duan, S.-J. Lai, and J. Huang, "A novel integration method for weak singularity arising in two-dimensional scattering problems," *Antennas and Propagation, IEEE Transactions on*, vol. 58, no. 8, pp. 2725 –2731, 2010.

[3] D. Taylor, "Accurate and efficient numerical integration of weakly singular integrals in galerkin efie solutions," *Antennas and Propagation, IEEE Transactions on*, vol. 51, no. 7, pp. 1630 – 1637, 2003.

[4] M. Khayat and D. Wilton, "Numerical evaluation of singular and near-singular potential integrals," *Antennas and Propagation, IEEE Transactions on*, vol. 53, no. 10, pp. 3180 – 3190, 2005.

[5] R. Graglia, "On the numerical integration of the linear shape functions times the 3-d green's function or its gradient on a plane triangle," *Antennas and Propagation, IEEE Transactions on*, vol. 41, no. 10, pp. 1448 –1455, Oct. 1993.

[6] L. Rossi and P. J. Cullen, "On the fully numerical evaluation of the linear-shape function times the 3d green's function on a plane triangle," *IEEE Transactions on Microwave Theory and Techniques*, vol. 47, no. 4, pp. 398–402, 1999.

[7] C. A. Balanis, *Advanced Engineering Electromagnetic*. New York, USA: John Wiley & sons, Inc., 1989.

[8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed. New York: Dover, 1964.

[9] W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw-Hill, 1986.

[10] A. Jeffrey and D. Zwillinger, Eds., *Table of Integrals, Series, and Products*. Academic Press, 1980, vol. Fourth. [Online]. Available: http://www.amazon.com/dp/012294755X

[11] R. Harrington, *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.