# The 8×8 DCT: the gory details

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## 1 The 8×8 DCT

The 2-D DCT of an 8×8 block of pixels $f(i, j)$ is given by:

$$F(u,v) = \frac{\Lambda(u)\Lambda(v)}{4} \sum_{i=0}^{7} \sum_{j=0}^{7} \cos\frac{(2 \times i + 1)\pi u}{16} \cos\frac{(2 \times j + 1)\pi v}{16} f(i,j)$$

where:

$$\Lambda(\epsilon) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \epsilon = 0 \\ 1 & \text{otherwise} \end{cases}$$

Since the DCT is a separable transform it can be computed using two 1-D DCTs:

$$G(i,v) = \frac{1}{2} \sum_{j=0}^{7} \Lambda(v) \cos\frac{(2 \times j + 1)\pi v}{16} f(i,j)$$

$$F(u,v) = \frac{1}{2} \sum_{i=0}^{7} \Lambda(u) \cos\frac{(2 \times i + 1)\pi u}{16} G(i,v)$$

where $\Lambda(\epsilon)$ is defined as above.

## 2 Implementing the 8×8 DCT

A straightforward implementation of the $8 \times 8$ DCT in software requires only simple marix multiplications. To see this we start from 2 1-D DCTs:

$$G(i,v) = \frac{1}{2} \sum_{j=0}^{7} \Lambda(v) \cos\frac{(2 \times j + 1)\pi v}{16} f(i,j)$$

$$F(u,v) = \frac{1}{2}\sum_{i=0}^{7}\Lambda(u)\cos\frac{(2\times i+1)\pi u}{16}G(i,v)$$

where:

$$\Lambda(\epsilon) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \epsilon = 0 \\ 1 & \text{otherwise} \end{cases}$$

When $v = 0$:

$$\begin{aligned} G(i,0) &= \frac{1}{2}\sum_{j=0}^{7}\frac{1}{\sqrt{2}}f(i,j) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}f(i,0) + \frac{1}{\sqrt{2}}f(i,1) + \ldots\right) \\ &= \frac{1}{2}\left[\begin{array}{ccc} f(i,0) & f(i,1) & \ldots \end{array}\right]\left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \vdots \end{array}\right] \\ &= \left[\begin{array}{ccc} f(i,0) & f(i,1) & \ldots \end{array}\right]\left[\begin{array}{c} \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \vdots \end{array}\right] \end{aligned}$$

When $v \neq 0$:

$$\begin{aligned} G(i,v) &= \frac{1}{2}\left(\cos\frac{\pi v}{16}f(i,0) + \cos\frac{3\pi v}{16}f(i,1) + \cos\frac{5\pi v}{16}f(i,2) + \ldots\right) \\ &= \frac{1}{2}\left[\begin{array}{cccc} f(i,0) & f(i,1) & f(i,2) & \ldots \end{array}\right]\left[\begin{array}{c} \cos\frac{\pi v}{16} \\ \cos\frac{3\pi v}{16} \\ \cos\frac{5\pi v}{16} \\ \vdots \end{array}\right] \end{aligned}$$

When $i \geq 0, v \geq 0$:

$$G(0-7,v) = \left[\begin{array}{ccc} f(0,0) & \ldots & f(0,7) \\ \vdots & \ldots & \vdots \\ f(7,0) & \ldots & f(7,7) \end{array}\right]\left[\begin{array}{c} \frac{1}{2}\cos\frac{\pi v}{16} \\ \frac{1}{2}\cos\frac{3\pi v}{16} \\ \frac{1}{2}\cos\frac{5\pi v}{16} \\ \vdots \end{array}\right]$$

$$G(0-7,0-7) = \begin{bmatrix} f(0,0) & \dots & f(0,7) \\ \vdots & \dots & \vdots \\ f(7,0) & \dots & f(7,7) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{2}\cos\frac{\pi}{16} & \frac{1}{2}\cos\frac{2\pi}{16} & \dots \\ \frac{1}{\sqrt{8}} & \frac{1}{2}\cos\frac{3\pi}{16} & \frac{1}{2}\cos\frac{6\pi}{16} & \dots \\ \frac{1}{\sqrt{8}} & \frac{1}{2}\cos\frac{5\pi}{16} & \frac{1}{2}\cos\frac{10\pi}{16} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

When $u = 0$:

$$\begin{aligned} F(0,v) &= \frac{1}{2}\sum_{i=0}^{7}\frac{1}{\sqrt{2}}G(i,v) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}G(0,v) + \frac{1}{\sqrt{2}}G(1,v) + \dots\right) \\ &= \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \dots \end{bmatrix} \begin{bmatrix} G(0,v) \\ G(1,v) \\ \dots \end{bmatrix} \end{aligned}$$

When $u \neq 0$:

$$\begin{aligned} F(u,v) &= \frac{1}{2}\left(\cos\frac{\pi u}{16}G(0,v) + \cos\frac{3\pi u}{16}G(1,v) + \cos\frac{5\pi u}{16}G(2,v) + \dots\right) \\ &= \begin{bmatrix} \frac{1}{2}\cos\frac{\pi u}{16} & \frac{1}{2}\cos\frac{3\pi u}{16} & \frac{1}{2}\cos\frac{5\pi u}{16} & \dots \end{bmatrix} \begin{bmatrix} G(0,v) \\ G(1,v) \\ G(2,v) \\ \vdots \end{bmatrix} \end{aligned}$$

When $u \geq 0, v \geq 0$:

$$F(0-7,v) = \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \dots \\ \frac{1}{2}\cos\frac{\pi}{16} & \frac{1}{2}\cos\frac{3\pi}{16} & \frac{1}{2}\cos\frac{5\pi}{16} & \dots \\ \frac{1}{2}\cos\frac{2\pi}{16} & \frac{1}{2}\cos\frac{6\pi}{16} & \frac{1}{2}\cos\frac{10\pi}{16} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} G(0,v) \\ G(1,v) \\ G(2,v) \\ \vdots \end{bmatrix}$$

$$F(0-7,0-7) = \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \dots \\ \frac{1}{2}\cos\frac{\pi}{16} & \frac{1}{2}\cos\frac{3\pi}{16} & \frac{1}{2}\cos\frac{5\pi}{16} & \dots \\ \frac{1}{2}\cos\frac{2\pi}{16} & \frac{1}{2}\cos\frac{6\pi}{16} & \frac{1}{2}\cos\frac{10\pi}{16} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} G(0,0) & \dots & G(7,0) \\ \vdots & \vdots & \vdots \\ G(7,0) & \dots & G(7,7) \end{bmatrix}$$

It should now be clear that:

$$\begin{aligned} G &= f \times C^T \\ \text{and } F &= C \times G \end{aligned}$$

where $f$ is the $8 \times 8$ input block of pixels, $G$ is a temporary matix in the implementation, $F$ is the $8 \times 8$ output block of DCT coefficients, and $C$ is the transform matrix which need only be calculated once in any given implementation.

$$C = \begin{bmatrix} \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \cdots \\ \frac{1}{2}\cos\frac{\pi}{16} & \frac{1}{2}\cos\frac{3\pi}{16} & \frac{1}{2}\cos\frac{5\pi}{16} & \cdots \\ \frac{1}{2}\cos\frac{2\pi}{16} & \frac{1}{2}\cos\frac{6\pi}{16} & \frac{1}{2}\cos\frac{10\pi}{16} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

## 2.1 The DC Coefficient

$$\begin{aligned} F(0,0) &= \frac{1}{2}\sum_{i=0}^{7}\frac{1}{\sqrt{2}}G(i,0) \\ &= \frac{1}{2}\frac{1}{\sqrt{2}}\left(G(0,0)+G(7,0)\right) \\ &= \frac{1}{2}\frac{1}{\sqrt{2}}\left(\left(\frac{1}{\sqrt{8}}f(0,0)+\ldots+\frac{1}{\sqrt{8}}f(0,7)\right)+\ldots+\left(\frac{1}{\sqrt{8}}f(7,0)+\frac{1}{\sqrt{8}}f(7,0)\right)\right) \\ &= \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{8}}\left(f(0,0)+\ldots+f(0,7)+\ldots+f(7,0)+\ldots+f(7,7)\right) \\ &= \frac{1}{8}\sum_{i,j=0}^{7} f(i,j) \end{aligned}$$

# 3 Implementing the Inverse DCT

Since the DCT is an *orthogonal* transform, we know that $C^{-1} = C^T$, which means $CC^T = C^TC = I$. Thus in order to find the *inverse* DCT trandform:

$$\begin{aligned} F &= CfC^T \\ C^TF &= C^TCfC^T \\ C^TFC &= C^TCfC^TC \\ C^TFC &= (C^TC)f(C^TC) \\ f &= C^TFC \end{aligned}$$