5. Fourier Analysis of Signals

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5.1 Introduction

- **In this chapter…**
  - Cover the necessary groundwork to understand *transform-based coding of signals*
    - As employed in image/video compression standards (JPEG, MPEG, H.26x, etc.)

- **Specifically…**
  - Examine the various mathematical representations available for characterising a signal

    …focussing in particular on Fourier's approach!

  - E.g. show how signal can be *decomposed* → i.e. expressed in terms of a linear sinusoidal-based expansion…
    - Discuss in detail the properties of such an expansion
      - Why it is useful
      - How it may be exploited to our advantage

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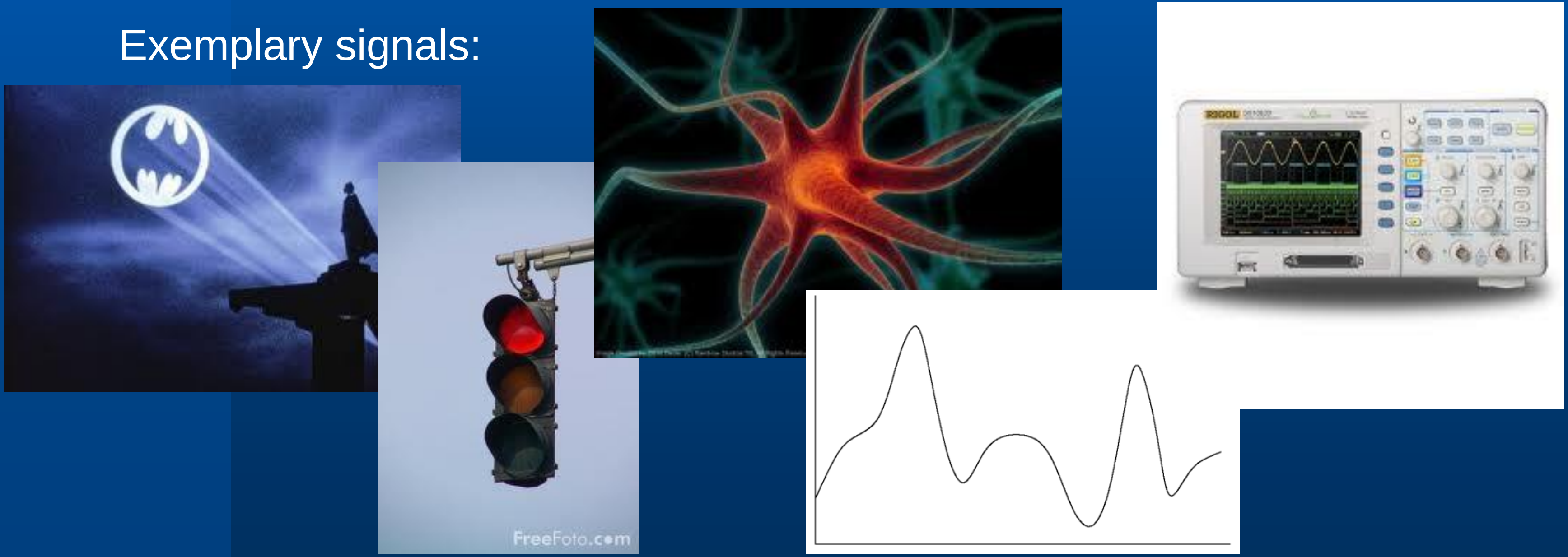
5.2.1 Digital & Analogue Signals

- **Definition of a signal…**
  - In some information technology contexts, a signal is simply "that which is sent or received," thus including both the carrier and the data together.
  - Measurement of some physical quantity over time and/or space

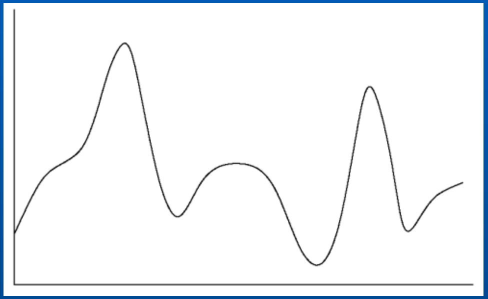
Exemplary signals:

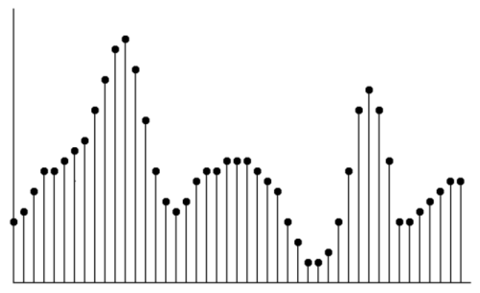


5.2.1 Digital & Analogue Signals

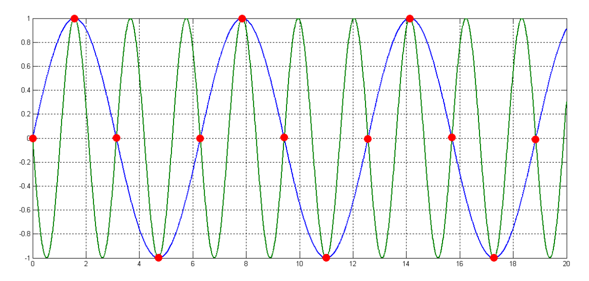
- **Sampling a signal in the time/space domain…**

  → yields a ***digital*** signal → only defined at <u>discrete</u> points





- **How finely do we need to sample?**
  - Nyquist Theorem:
    - *Sampling rate must be at least twice the highest frequency of the original analogue signal for the digital representation to capture it faithfully*



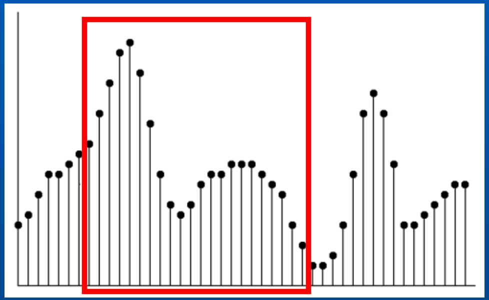
Green – original signal
Red – sampling points
Blue – reconstructed signal

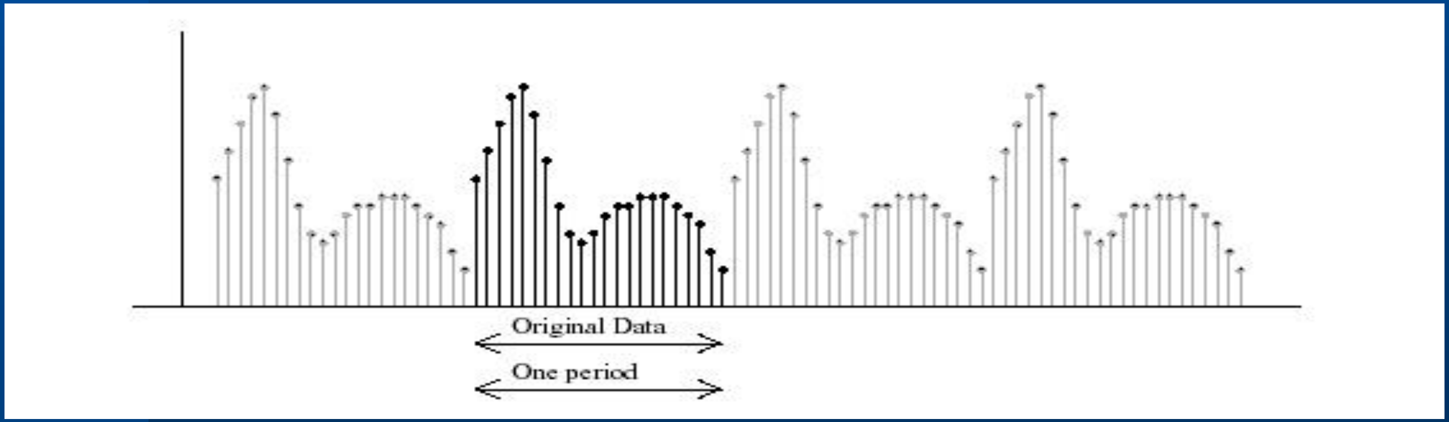
5.2.2 Constructing Periodicity

- **Fourier's Theorem (FT)**
  - *Any periodic function can be represented as a linear combination of sine and cosine functions…*
- **Problem: Real-world signals typically non-periodic**
- **Solution: Process small sub-sections one-at-a-time…**
  - i.e. consider sub-section as one period of a (synthesised) periodic function → then deploy Fourier's Theorem!
  - Hence the study of how to *artificially construct periodicity*

5.2.2 Constructing Periodicity (cont.)

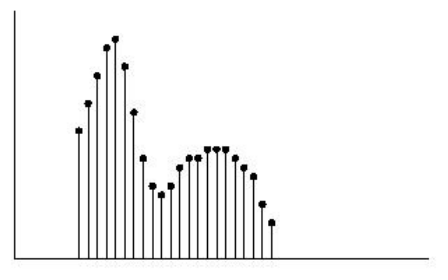
- **E.g. Applying FT to non-periodic signal (digital)…**
  - Approach: Piece-by-piece, use the data in the signal to create individual artificial periodic functions…
    - Select a small block of data
    - Consider it as one period of a periodic function
    - Apply FT to this artificial function
    - Move on to the next block of data

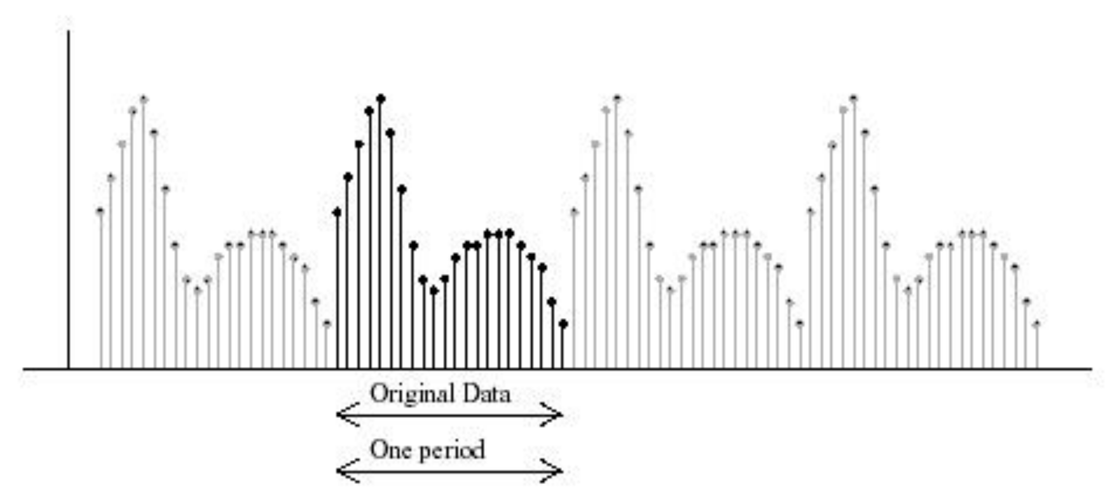




5.2.2 Constructing Periodicity (cont.)

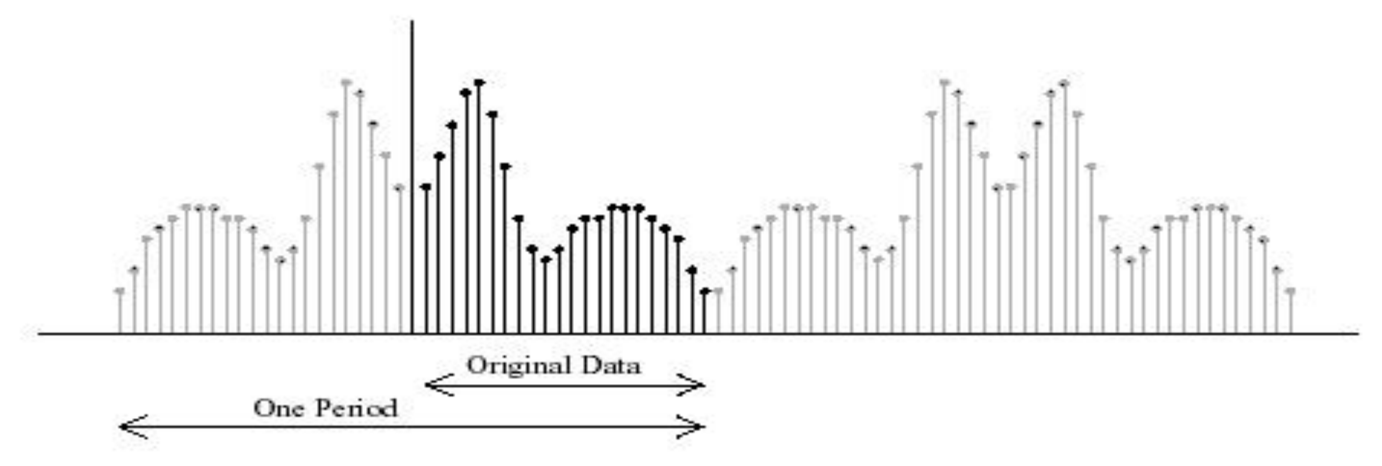
- **Approach1 (basic)**
  - Simple inifinite repetition of the selected block of data…





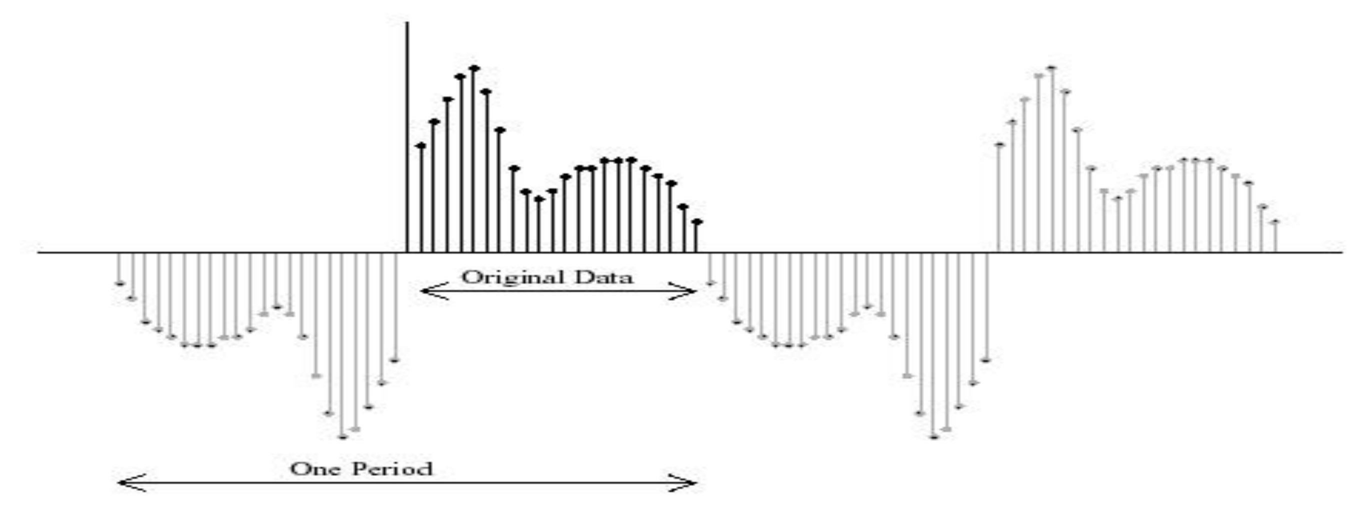
5.2.2 Constructing Periodicity (cont.)

- **Approach2 (constructing an even function)**
  - Double it up, place it back-to-back with itself, then repeat…
    - Constructed period twice as long as data block
    - If centred on origin → becomes an ‘even function’ : *f(x) = f(-x)*



5.2.2 Constructing Periodicity (cont.)

- **Approach3 (constructing an odd function)**
  - Place it back-to-back with the inverse of itself, then repeat…
    - Constructed period twice as long as data block
    - When centred on origin → becomes an 'odd function' : - *f(x)* = *f(-x)*



→ Usefulness of various approaches becomes clear later!

5.2.3 Comparing Functions

- **Recall, Dot Product (DP) enables us to compare two vectors…**
  - E.g. $A = (a_1, a_2, a_3)$ $B = (b_1, b_2, b_3)$…

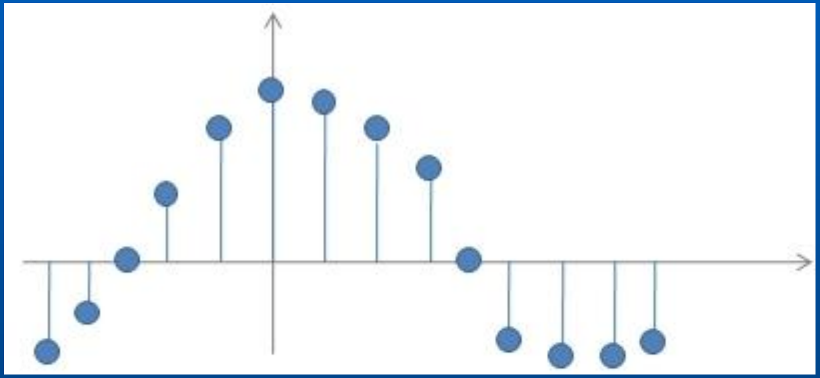
$$A \bullet B = a_1b_1 + a_2b_2 + a_3b_3$$

- **How can we interpret this as a *comparison* of A and B?**
  - If corresponding components (e.g. a_1 and b_1) are both positive/negative (i.e. both pointing in the same direction), given that they are multiplied, they will make a positive contribution to the sum

- **Zero DP?**
  - Equal amounts of (dis)similarity between A and B
  - Vectors are said to be '**orthogonal**'

5.2.3 Comparing Functions (cont.)

- **Comparing functions as opposed to vectors**
  - For *discrete* functions…
    - → same principle as DP!
    - i.e. can consider *N* sampled values of a (discrete) function as component values of an *N*-dimensional vector and use DP as before.



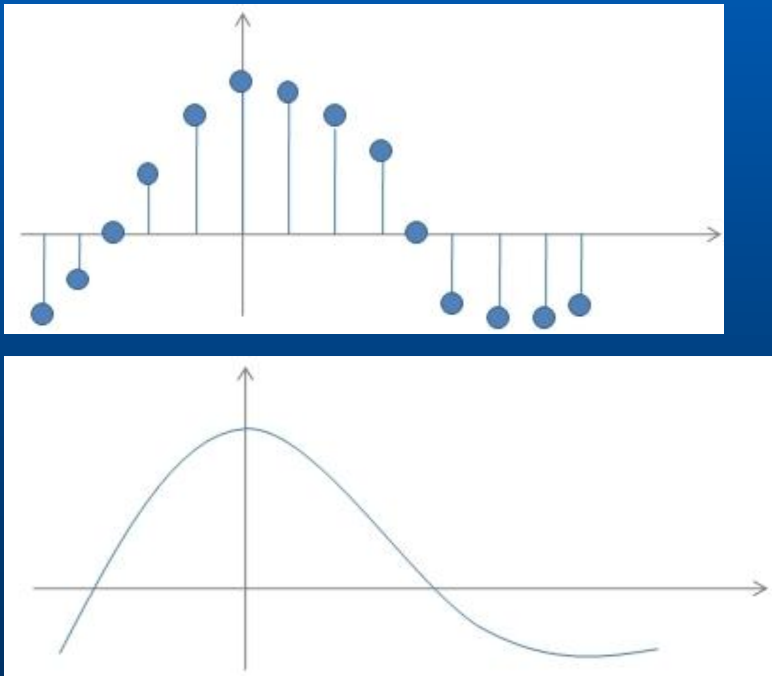
  - So, to *compare* two discrete functions: mutiply corresponding sample values and add the result

    → gives the 'similarity' measure

5.2.3 Comparing Functions (cont.)

- **Comparing functions as opposed to vectors (cont.)**
  - N.B. As sampling rate ↑ towards infinity → move towards cont. domain!
  - Process of multiplying corresponding values and adding becomes one of *integrating* the product of the two functions…
    - Known as finding the ‘**Inner Product**’ of two functions
    - E.g. the inner product of continuous functions *f(x)* and *g(x)* over the interval *[p, q]*…

$$< f, g >= \int_p^q f(x)g(x)dx$$



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5.3.2 Fourier's Theorem

- **Fourier's deduction:** Any *periodic* function may be expressed as a (infinite) linear combination of sines and cosines

- **Specifically, he formulated the following theorem…**
  - A periodic function *f(x)* - or a function defined over a finite interval [0, L] - which has a finite number of finite-size jump discontinuities over the interval of definition, can be expanded in the form…

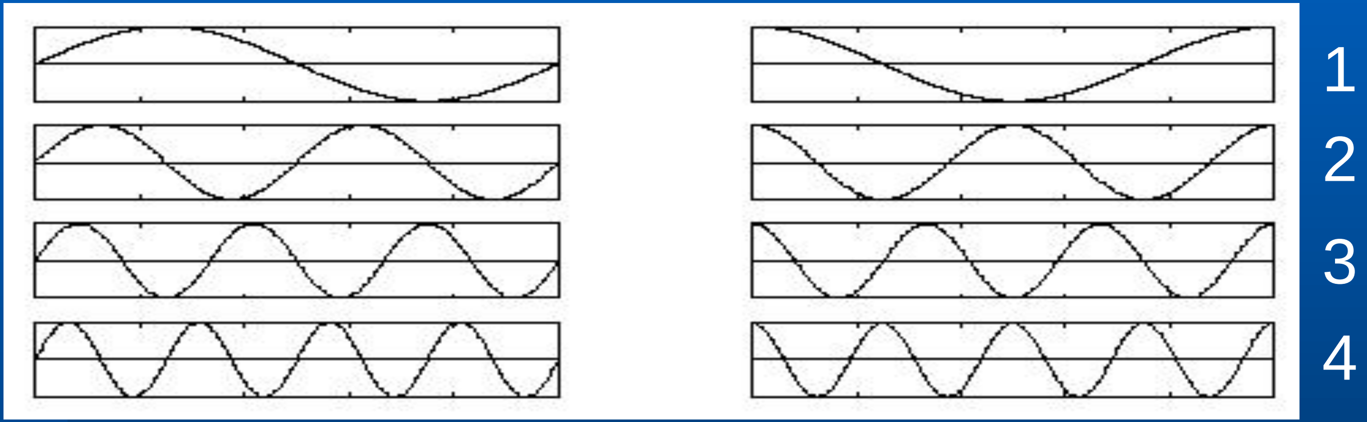
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

… where the weights a_n and b_n are called 'Fourier Coefficients'

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.2 Fourier’s Theorem (cont.)

- **The basis of Fourier’s Theorem:** Set of sine/cosine fns, which have an integer number of periods over the interval (**L**)…



  - **May be shown**: Every one of these functions orthogonal to every other function in the set
    - i.e. inner-products (similarity) = 0

$$\int_0^L \sin n\omega x \cos m\omega x dx = 0,$$
for all integers n and m

  - Importance of orthogonality of ‘integer cycle’ sines/cosines fundamental to Fourier analysis → see later

5.3.2 Fourier's Theorem (cont.)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

- **To analyse further → integrate both sides…**
  - Apart from the first term, each term on the RHS will involve evaluating the area under the graph of a sine/cosine function with an integer number (whole number) of cycles → obviously = 0
  - So…

$$\int_0^L f(x)dx = \int_0^L \frac{1}{2}a_0 dx$$

OR

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

  - Hence, the first term $\frac{1}{2}a_0$ = the mean of *f(x)* over one period *L*

    → known as the **DC-coefficient**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.2 Fourier's Theorem (cont.)

- **It may be shown (via further mathematical manipulation)…**

$$a_n = \frac{2}{L}\int_0^L f(x) \cos n\omega x dx$$

AND

$$b_n = \frac{2}{L}\int_0^L f(x) \sin n\omega x dx$$

- **So, together with**

$$a_0 = \frac{2}{L}\int_0^L f(x) dx$$

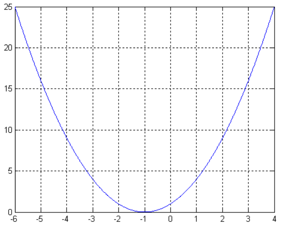
… gives us our **complete set of formulae for Fourier Series expansion**

i.e. in applying Fourier analysis…

- Base sines/cosines known (to hand)
- Corresponding coefficients (a_n and b_n) initially unknown and need to be calculated

Data Transformations

- **Applying Fourier Transform to continuous (analogue) signals**
  - Consider continuous function… (e.g.)



$$f(x) = x^2 + 2x + 1 \ :\ x \ \varepsilon \ \Re$$

i.e. for all real values *x*; *f(x)* has corresponding value defined (property of cont. function)

- - - Equation represents infinite no. of mappings from individual domain values *x* to individual range values *f(x)*
  - **Q/** *How could we describe this function without equation?*
    - ***A/*** One way: state all values at an infinite no. of points in its domain!
  - **Transforming function via Fourier Transform (FT)…**
    - Still need an infinite no. of sine/cosine terms to exactly describe the original function
    - However, relatively small number usually approximates well ☺
      - Imagine trying to describe a cont. function by small subset of its domain/range!

Data Transformations (cont.)

- **Applying Fourier Transform to discrete (digital) signals**
  - FT can also be extended to the discrete (sampled) signal case
    - i.e. where original signal is sampled at particular points
      (→ sine/cosine basis functions sampled at the same points)
  - Note the symmetry, wheen applying FT to a discrete signal…
    → E.g. **16** data samples in the original domain
    →**16** sine/cosine functions to exactly represent it (8 each)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.2 Fourier’s Theorem (cont.)

- **Other representations of the Fourier transform:**
  - Fourier Transform for continuous functions (by using Euler’s formula):

$$F(j\omega) = \int_{0}^{\infty} f(t) e^{-j\omega t}$$

$$e^{jn\omega x} = \cos(n\omega x) + j \sin(n\omega x)$$

  - Discrete Fourier Transform:

$$F(k) = \sum_{n=0}^{M-1} f(n) \exp\left(-j\frac{2\pi}{M}kn\right)$$

  - Fast Fourier Transform (an algorithm for computing DFT):

$$F(k) = \sum_{n=0}^{M-1} f(n) \exp\left(-j\frac{2\pi}{M}kn\right)$$

Data Transformations

- **Note, transform-based coding…**
  - Involves transforming data from <u>one domain into another</u> (e.g. mapping space/time values → values representing frequency)

Transform coding in general (where *K* – transformation kernel):

$$Tf(u) = \int_{t1}^{t2} K(t,u)\, f(t)\, dt$$

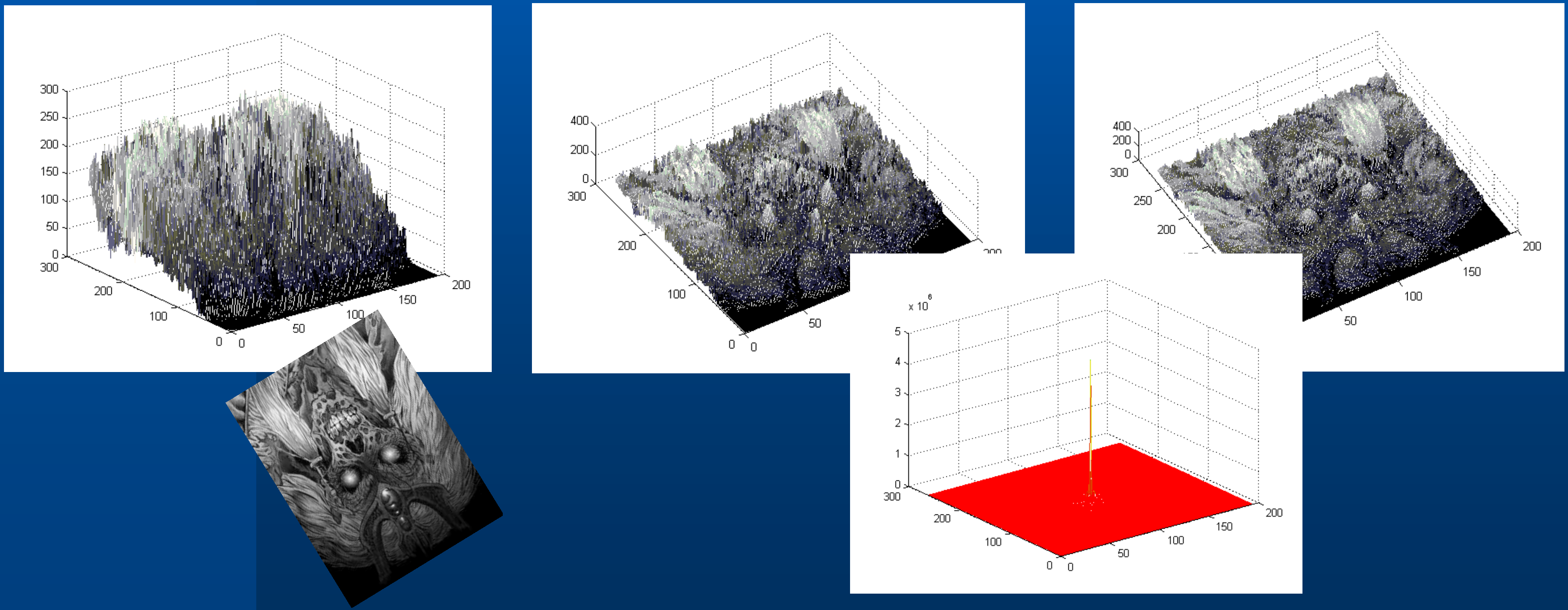
$$f(t) = \int_{u1}^{u2} K^{-1}(u,t)\, (Tf(u))\, du$$

Transform	**K**	**t1**	**t2**	**K-1**	**u1**	**u2**
Fourier	$\frac{e^{-iut}}{\sqrt{2\pi}}$	$-\infty$	∞	$\frac{e^{+iut}}{\sqrt{2\pi}}$	$-\infty$	∞
Laplace	e^{-ut}	$-\infty / 0$	∞	$\frac{e^{+ut}}{2\pi i}$	$c - j\infty$	$c + j\infty$
Mellin	t^{u-1}	0	∞	$\frac{t^{-u}}{2\pi i}$	$c - j\infty$	$c + j\infty$

Data Transformations (cont.)

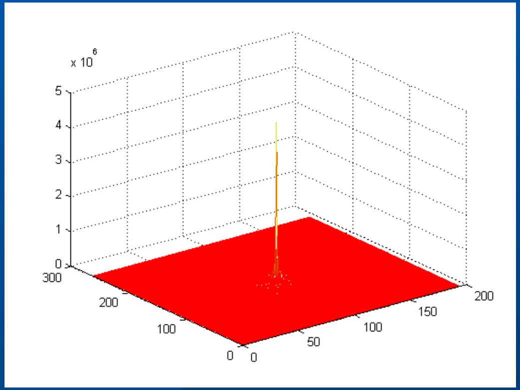
- **Considering image data…**
  - Image data = set of pixel ‘weights’ (intensities) in spatial domain
  - Transforming to frequency domain…

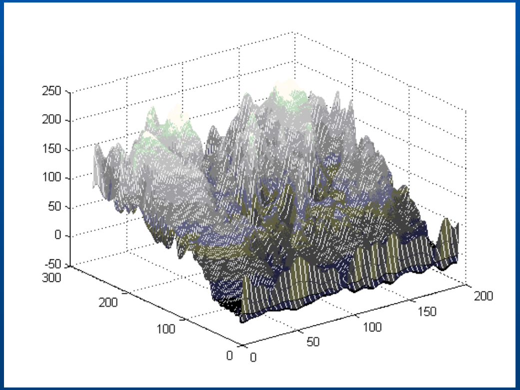
    → obtain corresponding set of ‘weights’ representing intensity of various frequency components (rate of pixel intensity change) in the data

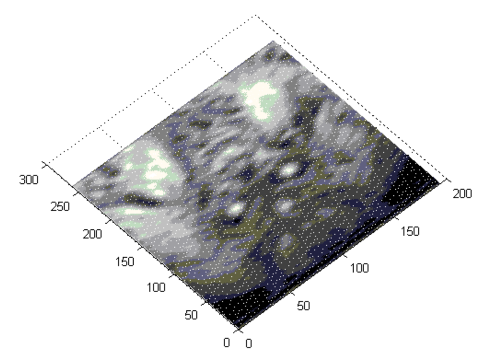


Data Transformations (cont.)

- **Considering image data…**
  - Turns out: Alternative (freq) representation typically allows us to see expendable bits ☺
    - Hence, a basis for compression!







N.B. A very powerful transform: Fourier’s Theorem (FT)

- *Any periodic function can be represented as an infinite linear combination of sine and cosine functions…*

2-D aliasing

- **How finely do we need to sample?**
  - Nyquist Theorem:
    - *Sampling rate must be at least twice the highest frequency of the original analogue signal for the digital representation to capture it faithfully*

Original image

Aliasiased image

5.3.3 Fourier Series Expansions

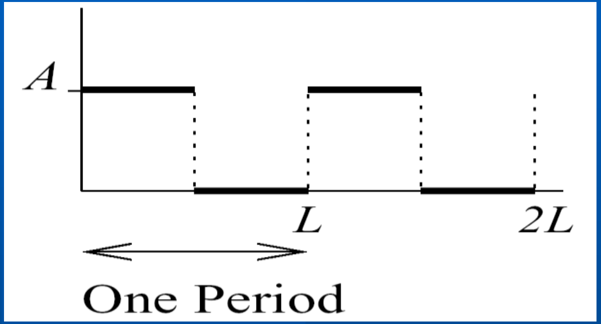
- **To continue the examination of Fourier theory…**
  - Look at some particular fundamental functions

    → examine their corresponding Fourier Series expansions
  - Objective: To learn something about the properties of the expansions from these examples
    - E.g. how they change, depending on the nature of the input signal being analysed

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example A: Positively-Biased Rectangular Waveform, period L**



$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\omega x dx$$

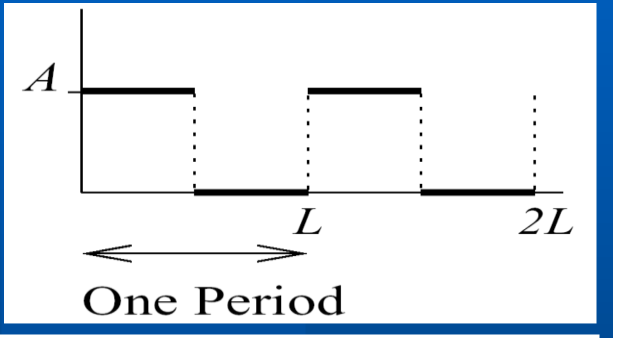
$$a_0 = \frac{2}{L}\int_0^L f(x)dx =$$
$$= \frac{2}{L}\int_0^{L/2} Adx + \frac{2}{L}\int_{L/2}^{L} 0dx =$$
$$= \frac{2}{L}Ax\,\Big|_0^{L/2} = \frac{2}{L}\frac{AL}{2} = A$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\omega x)dx =$$
$$= \frac{2}{L}\left(\int_0^{L/2} A\cos(n\omega x)dx + \int_{L/2}^{L} 0\cos(n\omega x)dx\right) =$$
$$= \frac{2A}{L}\int_0^{L/2}\cos(n\omega x)dx = \frac{2A}{Ln\omega}\sin(n\omega x)\,\Big|_0^{L/2} =$$
$$= \frac{2A}{Ln\omega}\sin\left(\frac{n\omega L}{2}\right) = \left\langle \omega = \frac{2\pi}{L}\right\rangle =$$
$$= \frac{A}{n\pi}\sin(n\pi) = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example A: Positively-Biased Rectangular Waveform, period L**



$$b_n = \frac{2}{L}\int_0^L f(x)\sin n\omega x dx$$

$$b_n = \frac{2A}{n\pi}$$ for odd *n*

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\omega x)dx =$$

$$= \frac{2}{L}\left(\int_0^{L/2} A\sin(n\omega x)dx + \int_{L/2}^{L} 0\sin(n\omega x)dx\right) =$$

$$= \frac{2A}{L}\int_0^{L/2}\sin(n\omega x)dx = -\frac{2A}{Ln\omega}\cos(n\omega x)\,|_0^{L/2} =$$

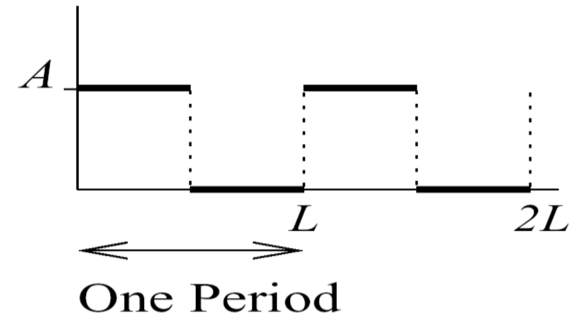
$$= -\frac{2A}{Ln\omega}\left(\cos\left(\frac{n\omega L}{2}\right) - \cos(0)\right) = \left\langle \omega = \frac{2\pi}{L} \right\rangle =$$

$$= -\frac{A}{n\pi}(\cos(n\pi) - 1) = \frac{A}{n\pi}(1 - \cos(n\pi))$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example A:** **Positively-Biased Rectangular Waveform, period L**



$$y = \begin{cases} A & \text{for } 0 \le x < L/2 \\ 0 & \text{for } L/2 \le x < L \end{cases}$$

  - F.S expansion…

Error: should be ‘2A’

$$a_0 = A, a_n = 0 \text{ for } n \neq 0$$

$$b_n = 0 \text{ for } n \text{ even}, b_n = \frac{2A}{n\pi} \text{dd}$$

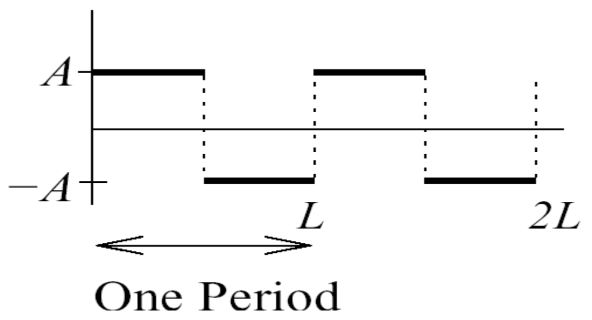
Typo!

$$y(x) = \frac{A}{2} + \frac{\boxed{2}}{\pi}\left(\sin \omega x + \frac{1}{3}\sin \boxed{h} 3\omega x + \frac{1}{5}\sin 5\omega x + \cdots\right)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example B: Zero-Average Rectangular Waveform, period L**



$$y = \begin{cases} A & \text{for } 0 \leq x < L/2 \\ -A & \text{for } L/2 \leq x < L \end{cases}$$

  - F.S expansion…

$$a_0 = 0,\, a_n = 0 \text{ for } n \neq 0$$

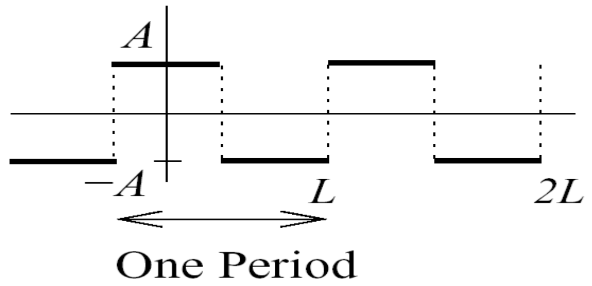
$$b_n = 0 \text{ for } n \text{ even}, b_n = \frac{4A}{n\pi} \text{ for } n \text{ odd}$$

$$y(x) = \frac{4A}{\pi}\left(\sin \omega x + \frac{1}{3}\sin 3\omega x + \frac{1}{5}\sin 5\omega x + \cdots\right)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example C: Zero-Average Rectangular Waveform, period L, with quarter-cycle phase shift**



$$y = \begin{cases} A & \text{for } 0 \le x < L/4 \\ -A & \text{for } L/4 \le x < 3L/4 \\ A & \text{for } 3L/4 \le x < L \end{cases}$$

  - F.S expansion…

$$a_0 = 0,\ a_n = \frac{4A}{\pi}\left[\ 1 \quad 0 \quad -\tfrac{1}{3} \quad 0 \quad \tfrac{1}{5} \quad \cdots\ \right]$$

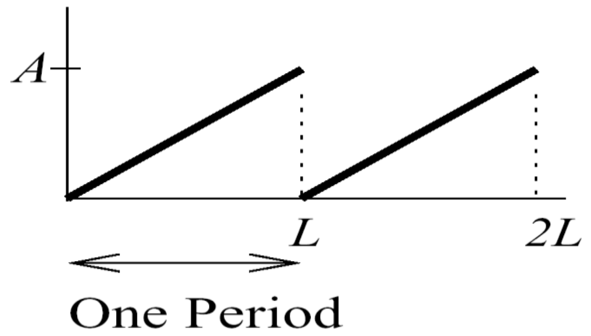
$$b_n = 0 \text{ for all } n$$

$$y(x) = \frac{4A}{\pi}\left(\cos\omega x - \frac{1}{3}\cos 3\omega x + \frac{1}{5}\cos 5\omega x + \cdots\right)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example D: Sawtooth Waveform, period L**



$$y = A\frac{x}{L} \text{ for } 0 \le x < L$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx =$$

$$= \frac{2}{L}\int_0^L \frac{A}{L}xdx = \frac{2A}{L^2}\frac{x^2}{2}\bigg|_0^L = A$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\omega x)dx =$$

$$= \frac{2}{L}\int_0^L \frac{Ax}{L}\cos(n\omega x)dx = \frac{2A}{L^2}\int_0^L x\cos(n\omega x)dx =$$

$$= \frac{2A}{L^2}\left(\frac{\cos(n\omega x)}{n^2\omega^2} + \frac{x\sin(n\omega x)}{n\omega}\right)\Bigg|_0^L =$$

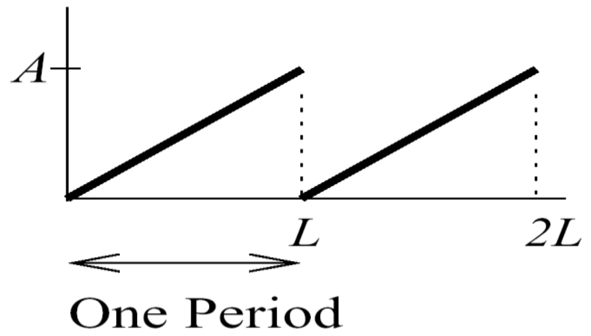
$$= \frac{2A}{L^2}\left(\frac{\cos(n\omega L)}{n^2\omega^2} + \frac{L\sin(n\omega L)}{n\omega} - \frac{1}{n^2\omega^2}\right) = \left\langle \omega = \frac{2\pi}{L} \right\rangle =$$

$$= \frac{A}{2\pi^2}\left(\frac{1}{4n^2\pi^2} - \frac{1}{4n^2\pi^2} + \frac{\sin(2n\pi)}{2\pi n}\right) = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example D: Sawtooth Waveform, period L**



$$y = A\frac{x}{L} \text{ for } 0 \le x < L$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\omega x)dx =$$

$$= \frac{2}{L}\int_0^L \frac{Ax}{L}\sin s(n\omega x)dx = \frac{2A}{L^2}\int_0^L x\sin(n\omega x)dx =$$

$$= \frac{2A}{L^2}\left(\frac{\sin(n\omega x)}{n^2\omega^2} - \frac{x\cos(n\omega x)}{n\omega}\right)\Bigg|_0^L =$$

$$= \frac{2A}{L^2}\left(\frac{\sin(n\omega L)}{n^2\omega^2} - \frac{L\cos(n\omega L)}{n\omega}\right) = \left\langle \omega = \frac{2\pi}{L} \right\rangle =$$

$$= -A\frac{1}{\pi n} = -\frac{A}{\pi n}$$

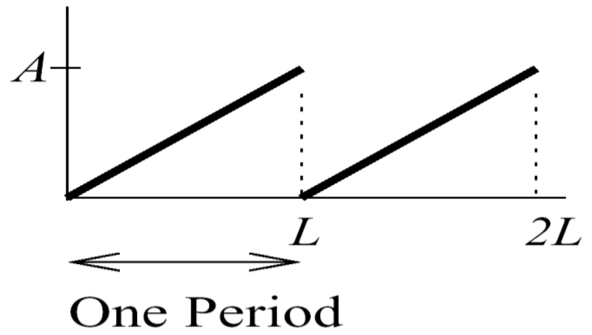
$$b_n = -\frac{A}{n\pi}$$

for all n

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.3 Fourier Series Expansions (cont.)

- **Example D: Sawtooth Waveform, period L**



$$y = A\frac{x}{L} \text{ for } 0 \leq x < L$$

  - F.S expansion…

$$a_0 = A, a_n = 0 \text{ for } n \neq 0$$

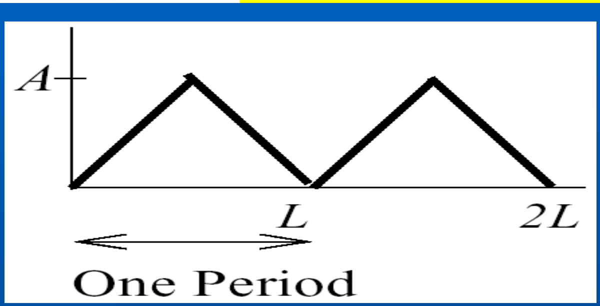
$$b_n = -\frac{A}{n\pi}$$

$$y(x) = \frac{A}{2} + \frac{A}{\pi}\left(-\sin\omega x - \frac{1}{2}\sin 2\omega x - \frac{1}{3}\sin 3\omega x - \cdots\right)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

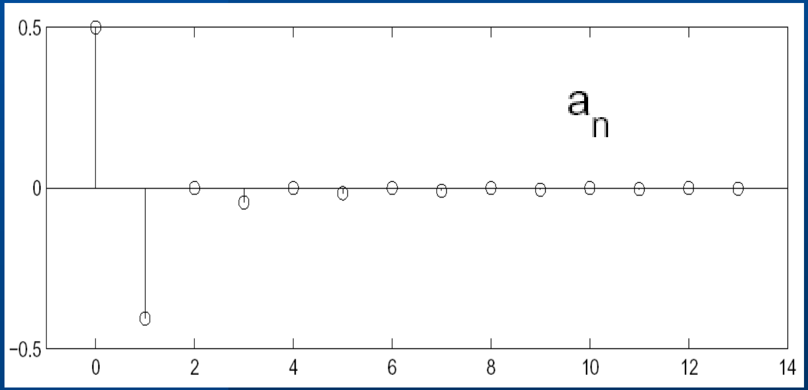
5.3.3 Fourier Series Expansions (cont.)

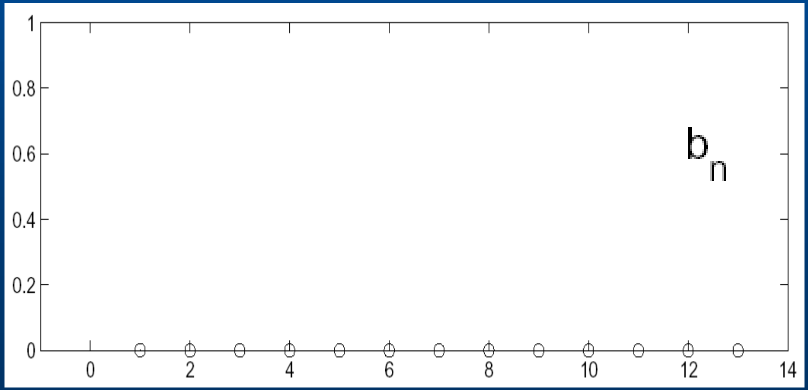
- **Example E: Triangular Waveform, Period L, Positive Bias**



$$y = \begin{cases} \frac{2Ax}{L} & \text{for } 0 \le x < L/2 \\ 2A(1 - \frac{x}{L}) & \text{for } L/2 \le x < L \end{cases}$$

First few coefficients of the corresponding F.S expansion…



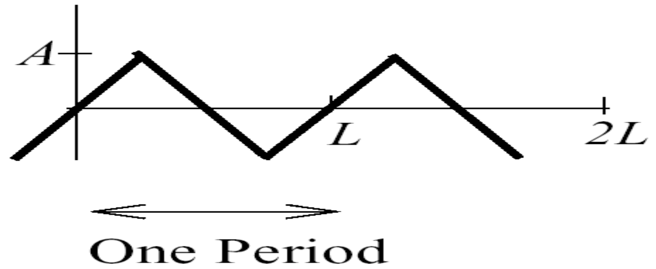


  - i.e. non-zero DC-coeff + only odd cosine terms (weighted by negative coeffients)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

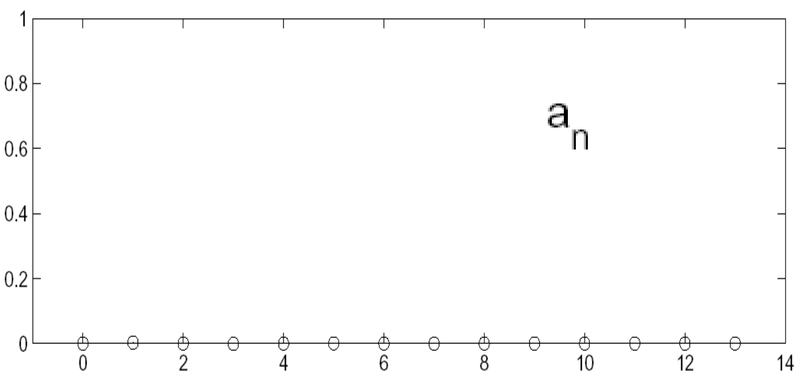
5.3.3 Fourier Series Expansions (cont.)

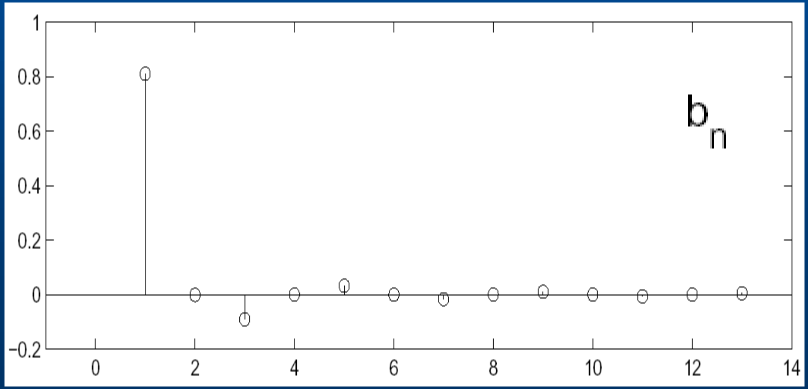
- **Example F: Triangular Waveform, Period L, Zero Bias, Odd Symmetry**



$$y = \begin{cases} 4A\frac{x}{L} & \text{for } 0 \leq x < L/4 \\ 2A(1 - \frac{2x}{L}) & \text{for } L/4 \leq x < 3L/4 \\ 4A(\frac{x}{L} - 1) & \text{for } 3L/4 \leq x < L \end{cases}$$

First few coefficients of the corresponding F.S expansion…





  - No DC-coeff ($a_0 = 0$) + only odd sine terms (with both positive + negative weights)

5.3.3 Fourier Series Expansions (cont.)

- **Examples:**

http://www.falstad.com/fourier/

WavePad Sound Editor

The FFT (example)

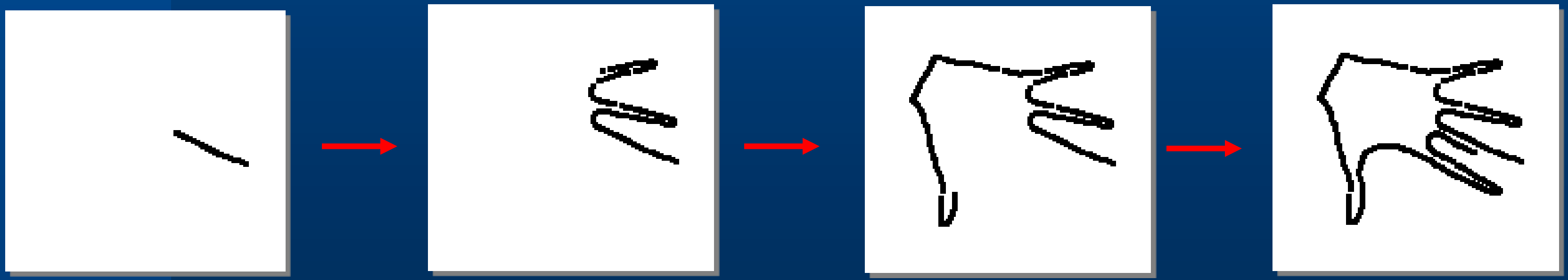
The Fast Fourier Transform given by the following equation has many advantages (easy and fast in implementation, etc.) and is frequently used in signal (image) processing algorithms.

$$F(k)=\sum_{n=0}^{M-1} f(n)\exp\left(-j\frac{2\pi}{M}kn\right)$$

Thanks to the usage of FFT in shape recognition process, it is possible to make it invariant to:

- object translation,
- scale,
- rotation,
- number of features that describe the object,
- choice of the starting point for contour tracking process.

In shape recognition process as an input for the algorithm we can treat the coordinates of successive object contour pixels.



The FFT (example)

In next step all the coordinates of pixels are changed to the complex numbers as follows:

$$z_n = x_n + j \cdot y_n$$

For contour given as a set of complex values we implement the FFT which results in vector of length *M* filled with Fourier coefficients:

$$\vec{F} = \left[F_0, F_1, F_2, \ldots, F_{M-3}, F_{M-2}, F_{M-1}\right]$$

Invariance of the FFT to the object location:

The location of the object has only influence to F0 coordinate of Fourier descriptor, forcing it to 0 we gain the invariance of FFT to location of the object in the image:

$$\vec{F} = \left[0, F_1, F_2, \ldots, F_{M-3}, F_{M-2}, F_{M-1}\right]$$

Invariance of the FFT to the object rotation:

The FFT invariance to object rotation can be accomplished by rotating the object by the following angle:

$$K_{rot} = -\frac{\arg\left(F_1\right) + \arg\left(F_{M-1}\right)}{2}$$

The FFT

Invariance of the FFT to the object scale:

Dividing all the coefficients of the Fourier descriptor by the power of the signal we can introduce the invariance of FFT to the object scale:

$$\vec{F}_{scale} = K_{scale} \cdot \vec{F}$$

$$K_{scale} = \frac{1}{\sqrt{|F_1|^2 + |F_2|^2 + |F_3|^2 + \ldots + |F_{M-3}|^2 + |F_{M-2}|^2 + |F_{M-1}|^2}}$$

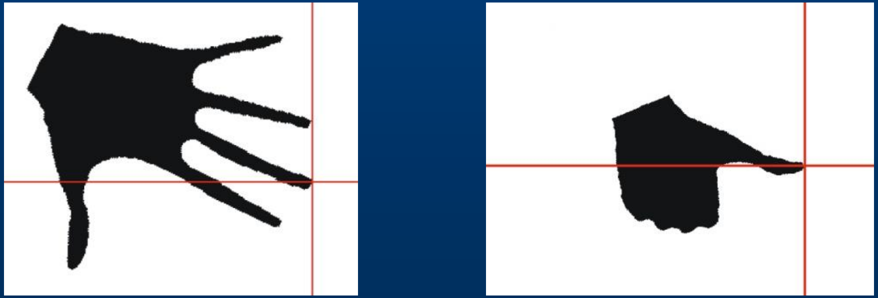
Invariance of the FFT to the number of coefficients describing the object:

The invariance of the FFT to the number of coefficients in FFT domain can be obtained by taking only most significant ones:

$$\vec{F} = \left[F_1, F_2, F_3, F_{M-3}, F_{M-2}, F_{M-1}\right]$$

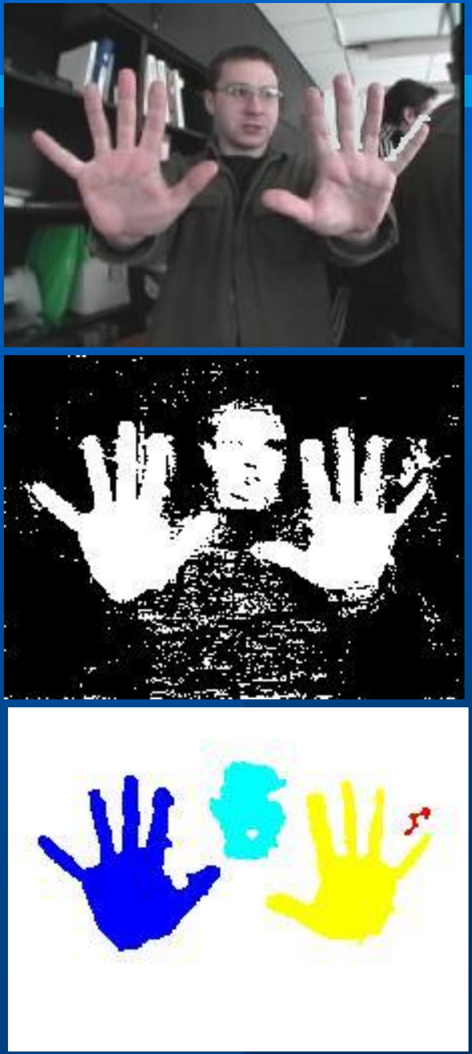
Invariance of the FFT to the contour starting point:

The algorithm starts always from the point located on the most right position of the object:

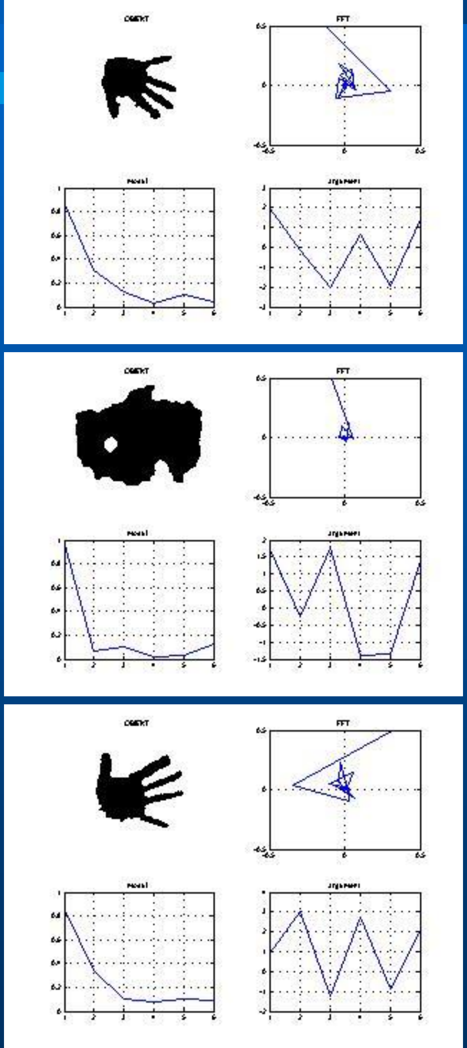


Example 1 (picture from video camera)

ORIGINAL, BINARY and SEGMENTED IMAGES



RECOGNIZED SHAPES AND THEIR FFT



ANN OPTPUT

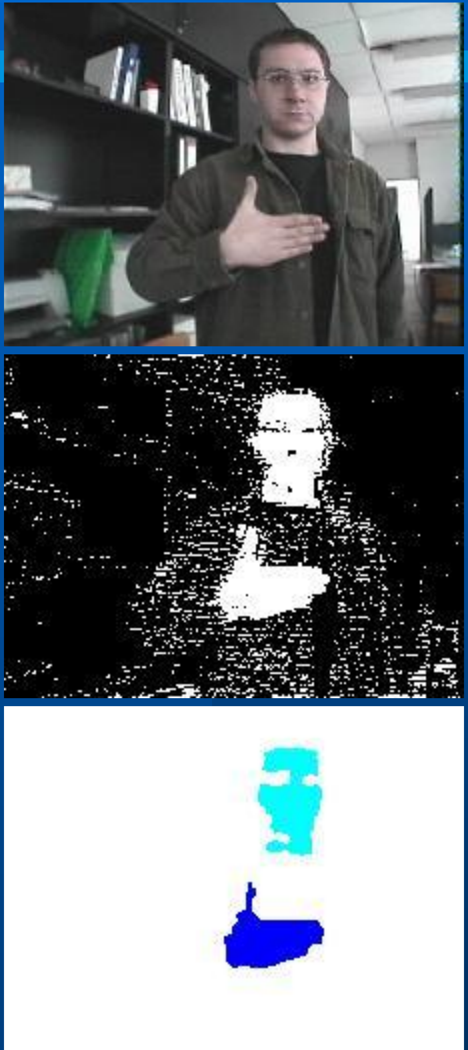
-2.11935	NO DEC.
1.53971	STOP
-1.03176	STOP
0.128173	WAIT
-2.02919	WAIT
-0.421842	OK
-1.77188	
-0.926982	

-1.79307	NO DEC.
-1.05637	STOP
-0.0923473	STOP
-0.356658	WAIT
-1.39729	WAIT
-0.768746	OK
-1.65126	OK
0.638963	HEAD

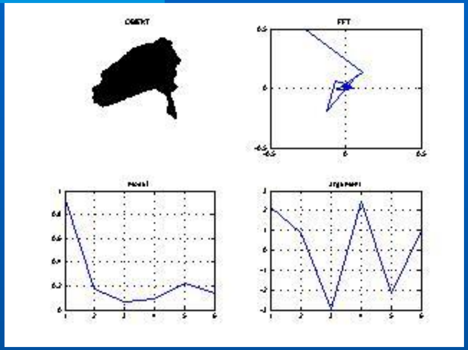
-0.318208	
-0.429639	
0.445989	STOP
-2.7277	WAIT
-0.306684	WAIT
-0.913592	OK
-0.759513	OK
-0.467065	HEAD

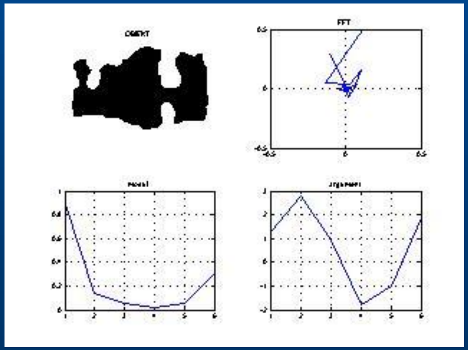
Example 2 (picture from video camera)

ORIGINAL, BINARY and SEGMENTED IMAGES



RECOGNIZED SHAPES AND THEIR FFT





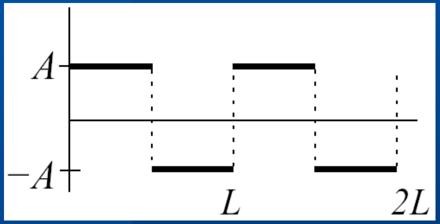
ANN OPTPUT

-1.40237	**NO DEC.**
-1.31203	**STOP**
-1.105	**STOP**
-0.299849	**WAIT**
-0.418272	**WAIT**
-0.410907	**OK**
-0.240858	**OK**
-0.683111	**HEAD**

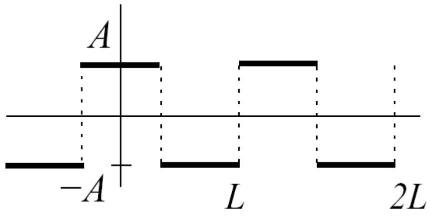
-0.744536	**NO DEC.**
-1.25713	**STOP**
-0.758346	**STOP**
-0.698045	**WAIT**
-1.97272	**WAIT**
-0.453919	**OK**
-0.85374	**OK**
-0.225581	**HEAD**

5.3.4 General Properties of F.S.

- **Property 1: Phase (spatial) shift of waveform**
  - Phase (spatial) shift does NOT directly affect the coeffs
    - Simply affects (correspondingly) the phase of the sine/cosine basis functions
  - Recall:



:
$$y(x) = \frac{4A}{\pi}\left(\sin\omega x + \frac{1}{3}\sin 3\omega x + \frac{1}{5}\sin 5\omega x + \cdots\right)$$



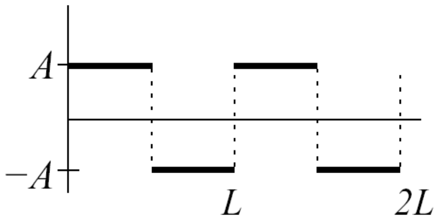
:
$$y(x) = \frac{4A}{\pi}\left(\cos\omega x - \frac{1}{3}\cos 3\omega x + \frac{1}{5}\cos 5\omega x + \cdots\right)$$

$$= \frac{4A}{\pi}\left(\sin\omega(x+\frac{\pi}{2}) + \frac{1}{3}\sin 3\omega(x+\frac{\pi}{2}) + \frac{1}{5}\sin 5\omega(x+\frac{\pi}{2}) + \cdots\right)$$

Same, but with 90° phase-shift

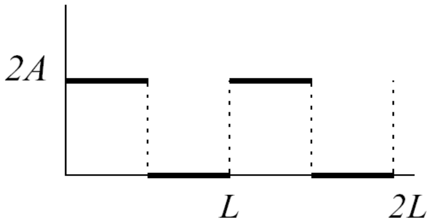
5.3.4 General Properties of F.S. (cont.)

- **Property 2: Amplitude shift of waveform**
  - Amplitude shift has NO effect on any of the coeffs, EXCEPT a_0
    - Shift in amplitude is simply added to the value of a_0 in the expansion
  - Recall the expansion for a rectangular waveform…



$$: \quad y(x) = \frac{4A}{\pi}\left(\sin \omega x + \frac{1}{3}\sin 3\omega x + \frac{1}{5}\sin 5\omega x + \cdots\right)$$

  - While…



$$: \quad y(x) = A + \frac{4A}{\pi}\left(\sin \omega x + \frac{1}{3}\sin 3\omega x + \frac{1}{5}\sin 5\omega x + \cdots\right)$$

5.3.4 General Properties of F.S. (cont.)

- **Property 3: Waveform even/odd symmetry**
  - Recall coefficient formulae…

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\omega x dx$$

AND

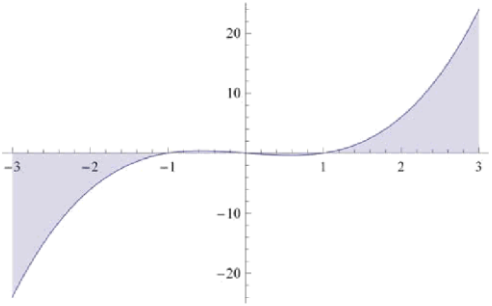
$$b_n = \frac{2}{L}\int_0^L f(x)\sin n\omega x dx$$

  - Note formulae correspond to calculating *inner-products*…
    - i.e. 'comparing' *f(x)* with the integer-cycle sine/cosine functions of the analysis
    - Note, these inner-products are carried out by multiplying the functions, and then integrating

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\omega x dx$$ $$b_n = \frac{2}{L}\int_0^L f(x)\sin n\omega x dx$$

5.3.4 General Properties of F.S. (cont.)

- **Property 3: Waveform Even/Odd Symmetry (cont.)**
  - So, calc a_n, b_n. ≡ multiplying *f(x)* with sines/cosines then integrating
  - Now, we know… sines → odd [- *f(x)* = *f(-x)*]
    cosines → even [*f(x)* = *f(-x)*]
  - Also know even/odd function product properties…
    - even function X even function = even function
    - odd function X odd function = even function
    - odd function X even function = odd function
  - Also know: Symmetric integral about the origin of odd function = 0



$$\int_{-L/2}^{L/2} f(x)dx = 0, \text{ for } f(x) \text{ odd.}$$

→ Can use above knowledge to explain the effect of even/odd waveform symmetry on Fourier Series expansions…

5.3.4 General Properties of F.S. (cont.)

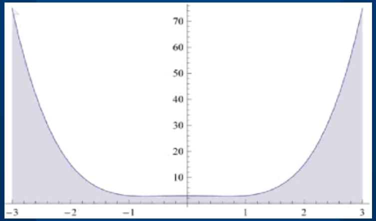
- **Property 3: Waveform Even/Odd Symmetry (cont.)**
  - 1st term (DC term) given as…

$$a_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\,dx$$

N.B. To simplify, the interval of definition of the functions taken to be: *[-L/2, L/2]*
(Still integrating over one period L)

→ This states: a_0 (the mean value of the overall expansion) is evaluated by a symmetric integral about the origin of *f(x)*…

  - The symmetric integral of an odd function about the origin = 0
    → hence a_0 for *f(x)* odd = 0
  - Not so for an EVEN function, where a_0 non-zero, and may be calculated as…



$$a_0 = \frac{4}{L}\int_{0}^{L/2} f(x)\,dx$$

i.e. integral of an even function from –A to +A is twice the integral from 0 to +A

5.3.4 General Properties of F.S. (cont.)

- **Property 3: Waveform Even/Odd Symmetry (cont.)**
  - The a_n coefficients involve integrals of the form…

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos n\omega x \, dx$$

  - Now if *f(x)* odd, since cosine even → their product is odd
    - i.e. we have a symmetric integral about the origin of an odd function

→ equates to zero → i.e. all $a_n = 0$

  - However, if *f(x)* even, since cosine even → their product is even
    - Hence the a_n are non-zero and may be calculated as…

$$a_n = \frac{4}{L}\int_{0}^{L/2} f(x)\cos n\omega x \, dx$$

i.e. integral of an even function from –A to +A is twice the integral from 0 to +A

5.3.4 General Properties of F.S. (cont.)

- **Property 3: Waveform Even/Odd Symmetry (cont.)**
  - The b_n coefficients involve integrals of the form…

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin n\omega x\,dx$$

  - Now if *f(x)* even, since sine odd → their product is odd
    - i.e. we have a symmetric integral about the origin of an odd function → equates to zero → i.e. all $b_n = 0$
  - However, if *f(x)* odd, since sine is odd → their product is even
    - Hence the b_n are non-zero and may be calculated as…

$$b_n = \frac{4}{L}\int_{0}^{L/2} f(x)\sin n\omega x\,dx$$

i.e. integral of an even function from –A to +A is twice the integral from 0 to +A

5.3.4 General Properties of F.S. (cont.)

- **Property 3: Waveform Even/Odd Symmetry (cont.)**
  - To summarise…

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

    - f(x) even : all b_n zero : expansion contains DC + cosine terms only
    - f(x) odd : all a_n zero : expansion contains sine terms only

Recall constructing periodicity from non-periodic functions…

- Mentioned 'doubling up' selected block of data towards creating odd/even periodic functions (with a period twice as long as the block of data). Why do this?
  - It means resulting artificial function will have either even/odd symmetry
    - Allows control of F.S. characteristics, i.e. which type of sinusoidal functions are present (very useful → see later!)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega x + \sum_{n=1}^{\infty} b_n \sin n\omega x$$

5.3.4 General Properties of F.S. (cont.)

$$f(x + \frac{L}{2}) = f(x)$$

- **Property 4: Waveform Translational Symmetry**
  - Definition: A waveform has *translational symmetry* if, after you slide it an appropriate amount (apply a particular translation), it looks exactly the same as before.
  - First → need to talk about even and odd *harmonics*…
    - Note: can regroup the terms of any Fourier expansion as follows…

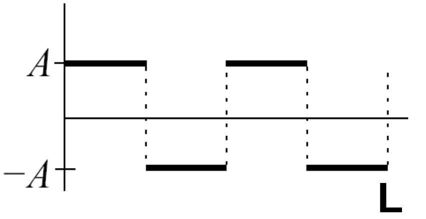
$$f(x) \quad = \quad \frac{1}{2}a_0 + (a_1 \cos \omega x + b_1 \sin \omega x) + (a_2 \cos 2\omega x + b_2 \sin 2\omega x) + \cdots$$

… i.e. those with functions of the <u>same frequency</u> are grouped together

- In this expression…
  - 1st non-zero frequency term = ‘fundamental’
  - 2nd non-zero frequency term = ‘2nd harmonic’ (an ‘even harmonic’)
  - 3rd non-zero frequenct term = ‘3rd harmonic’ (an ‘odd harmonic’), etc.

5.3.4 General Properties of F.S. (cont.)

- **Property 4: Waveform Translational Symmetry (cont.)**
  - Possible to show (not here!) that for periodic function *f(x)*…
    - Odd harmonics = 0 if it has *translational symmetry* over half its period…

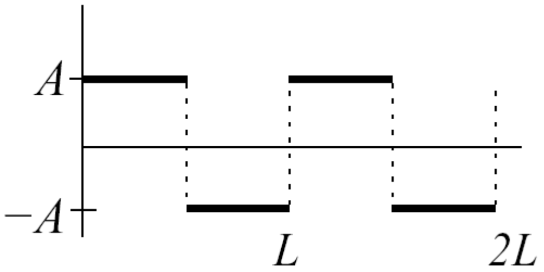


i.e. property $f(x+\frac{L}{2})=f(x)$ => expansion with no ‘odd terms’

    - Even harmonics = 0 if it has translational anti-symmetry over half its period…

i.e. property $f(x+\frac{L}{2})=-f(x).$ => expansion with no ‘even terms’

i.e. the same waveform but inverted



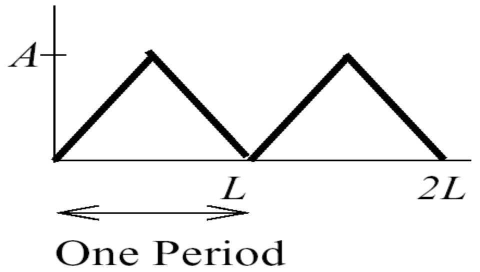
5.3.4 General Properties of F.S. (cont.)

- **Property 4:** **Waveform Translational Symmetry (cont.)**
  - In the examples shown (section 5.3.3)…
    - Waveforms that exhibit translational anti-symmetry when translated by L/2…
      - The rectangular waveform examples B & C
      - The triangular waveform examples E & F
        - Hence all have odd harmonics only in their respective F.S. expansions

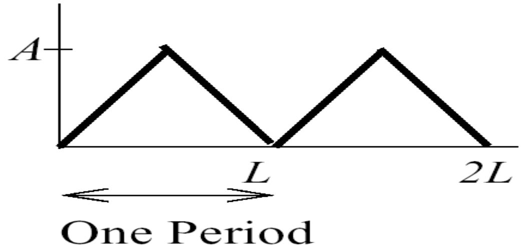
    - Only expansion to contain even harmonics → sawtooth waveform
      - Reason: Sawtooth does not exhibit translational symmetry (of any kind)
        - Hence, presence of both EVEN and ODD harmonics in its F.S. expansion

5.3.4 General Properties of F.S. (cont.)

- **Property 5: Waveform Discontinuities**
  - Property of F.S. expansions: As $n\uparrow$, (i.e. for higher harmonics)…
    - The Fourier coeffs a_n & b_n generally get increasingly smaller
    - Conclude: the higher harmonics are less significant in approximating *f(x)*
  - However, rate at which coeffs decrease with *n* (i.e. rate of '*expansion convergence*') depends on the characteristics of *f(x)*
  - May be shown…
    - Faster convergence related to periodic function *f(x)* not having any '**jump discontinuities'** - either within its period, or at the ends of the periodic interval
      - E.g. triangle waveform
  - In fact for a function with no jump discontinuities…
    - Convergence may be as fast as $1/n^2$



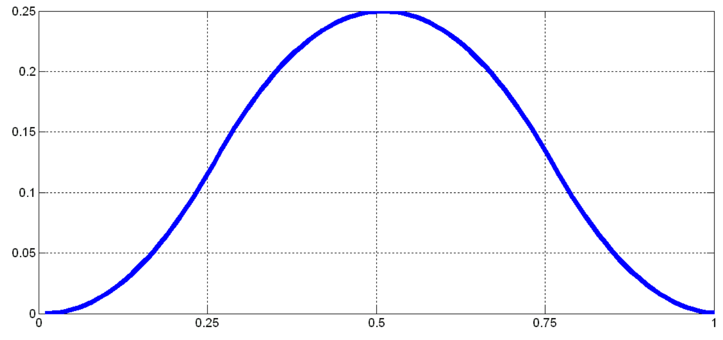
5.3.4 General Properties of F.S. (cont.)



- **Property 5: Waveform Discontinuities (cont.)**
  - May also be shown…
    - For *f(x)* with no jump discontinuities, convergence may be FASTER than $1/n^2$ when no jump discontinuities in the 1st derivative (slope) of *f(x)*
  - E.g. consider triangle waveform of example E (section 5.3.3)…
    - No jump discontinuities in orig. waveform, hence convergence may be as fast as $1/n^2$
    - However, there are two abrupt changes in direction (discontinuities in slope)…
      - 1. At the centre of the period
      - 2. At the end of the period
    - Hence unlikely that the expansion will converge any faster than $1/n^2$

5.3.4 General Properties of F.S. (cont.)

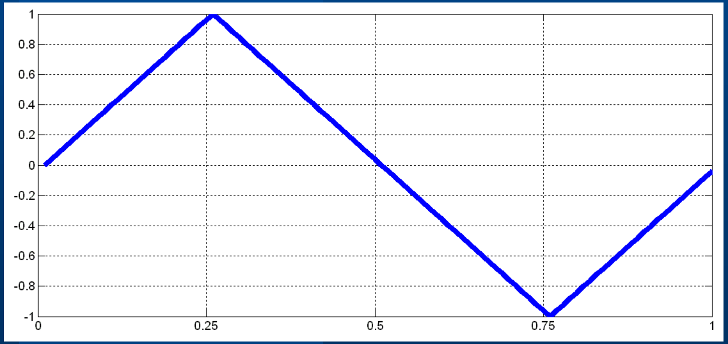
- **Property 5: Waveform Discontinuities (cont.)**
  - Another example, consider the function…



$$y = \begin{cases} 2x^2 & \text{for } 0 \le x < 0.25 \\ -2x^2 + 2x - 0.25 & \text{for } 0.25 \le x < 0.75 \\ 2x^2 - 4x + 2 & \text{for } 0.75 \le x < 1 \end{cases}$$

This is continuous (no jump discontinuities), hence <u>convergence may be as fast as *$1/n^2$*</u>

  - The first derivative of this is…



$$y = \begin{cases} 4x & \text{for } 0 \le x < 0.25 \\ -4x + 2 & \text{for } 0.25 \le x < 0.75 \\ 4x - 4 & \text{for } 0.75 \le x < 1 \end{cases}$$

suggests the function has abrupt changes (discontinuities) of slope at the changeover points of the branches - <u>unlikely</u> to converge <u>faster</u> than *$1/n^2$*

5.3.4 General Properties of F.S. (cont.)

- **Property 5: Waveform Discontinuities (cont.)**
  - Note, by examining subsequent derivatives → learn even more about convergence property of *f(x)*
  - E.g. 2nd derivative of previous function given by…

$$y = \begin{cases} 4 & \text{for } 0 \le x < 0.25 \\ -4 & \text{for } 0.25 \le x < 0.75 \\ 4 & \text{for } 0.75 \le x < 1 \end{cases}$$

… obvious jump discontinuities at the changeover points

→ Convergence is <u>unlikely</u> to be <u>faster</u> than $1/n^3$!

5.3.4 General Properties of F.S. (cont.)

- **Property 5: Waveform Discontinuities (cont.)**
  - Note, by examining subsequent derivatives → learn even more about convergence property of *f(x)*
  - E.g. 2nd derivative of previous function given by…

$$y = \begin{cases} 4 & \text{for } 0 \leq x < 0.25 \\ -4 & \text{for } 0.25 \leq x < 0.75 \\ 4 & \text{for } 0.75 \leq x < 1 \end{cases}$$

… obvious jump discontinuities at the changeover points

→ Convergence is unlikely to be faster than $1/n^3$!

→ Concludes the section on F.S. expansion properties

→ Usefulness will become clearer later, but first…

5.3.5 Complete Sets of Functions

- **Recall…**
  - Fourier coeffs = the results of comparing *f(x)* with the set of sines/cosines defined over the interval that have an integer no. of cycles in this interval

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\omega x dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin n\omega x dx$$

- **Also recall…**
  - This set of sines/cosines form an *orthogonal* (i.e. perpendicular) set
    - i.e. each comparison captures something unique about *f(x)* that is not captured by the others!

→ the similarity of *f(x)* and a particular sine/cosine function is not represented in any of the other comparisons

5.3.5 Complete Sets of Functions (cont.)

- **Note, Fourier states…**
  - "Any arbitrary function *f(x)* may be represented as a linear combination of this set of sines/cosines…"
    - Therefore, expansion must have the ability to capture every conceivable way in which *f(x)* may change
    - Hence, set of sine/cosine 'basis functions' represent what is known as a '*complete*' set of functions
- **In fact…**
  - Any *complete* set of functions can be used to represent *f(x)* in this way!

    → it doesn't necessarily have to be the set of integer-cycle sines/cosines used in Fourier's analysis

Overview

5.1 Introduction

5.2 Background to Fourier Expansions
- 5.2.1 Digital and Analogue Signals
- 5.2.2 Constructing Periodicity
- 5.2.3 Comparing Functions

5.3 Fourier Analysis of Periodic Functions
- (5.3.1 Comparing Sines and Cosines)
- 5.3.2 Fourier's Theorem
- 5.3.3 Fourier Series Expansions for Periodic Functions
- 5.3.4 General Properties of Fourier Series
- 5.3.5 Complete Sets of Functions

5.4 Fourier Analysis of Non-Periodic Functions
- 5.4.1 Fourier Transform
- 5.4.2 Half-Range & Quarter Range Series

5.4 Fourier Analysis: Non-Periodic Case

- **In practical real-world measurements…**
  - Not dealing with periodic signals!

- **Two ways of extending Fourier Analysis to <u>non-periodic functions</u>…**
  1. Consider *f(x)* to be periodic, but with infinitely long period
     - This approach develops into the *<u>Fourier Transform</u>*
  2. Select segment of *f(x)* (a ‘block’ of its data); consider as one cycle of a (synthesised) periodic function; deploy Fourier analysis; repeat

     → This approach develops into the DCT (crucial to our subject here)

5.4.1 The Fourier Transform

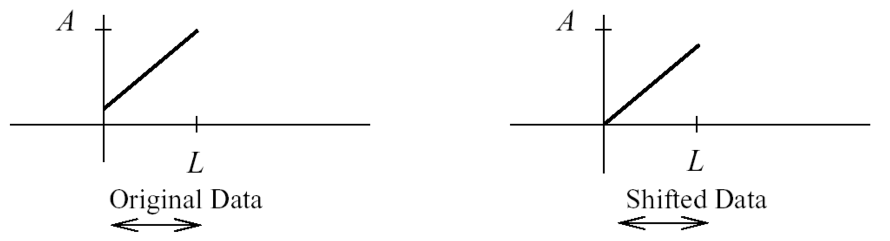
- **Fourier Transform in short…**
  - A technique to find the Fourier equivalent of a *non*-periodic function
    - Works by considering *f(x)* to be periodic, but with infinitely long period
  - Brief outline of its mathematical construction provided in the notes
    - For completeness only
    - Not pursued any further here
- For image & video compression…
  - More interested in the selection of data blocks approach

    → where the block of data is used to artifically construct periodicity

→ SEE NEXT…

5.4.2 Half/Quarter-Range Series

- **When constructing periodicity…**
  - Considering data block selected from *f(x)* as one period of a synthesised periodic function → known as constructing '*Full-Range Series*'



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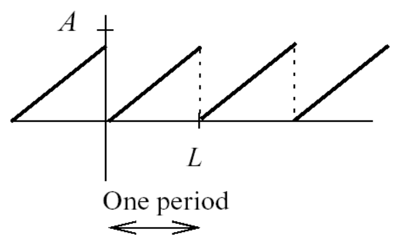
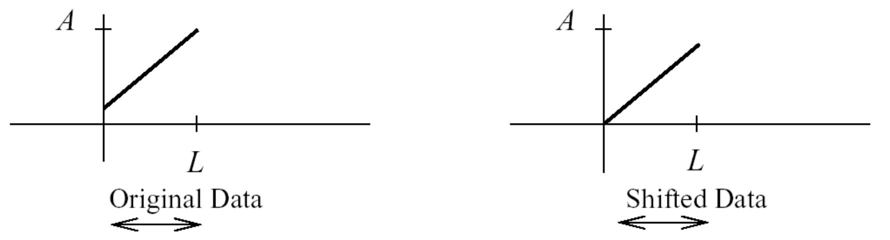


- **Implementing Full-Range Series means…**
  - Do not have much control over properties of derived series
  - If basis functions in the form of Complex exponentials (common)
    - → required to use Complex arithmetic when implementing

$$f(x) = \sum_{n=-\infty}^{+\infty} \left(c_n e^{jn\omega x} \right)$$

5.4.2 Half/Quarter-Range Series (cont.)

- **'*Half-Range Series'* approach…**
  - By considering selected data block as one half of the period of a periodic function (doubling it up to synthesise the period)…
  - Can create an *even* function... f(x) = *f(-x)*



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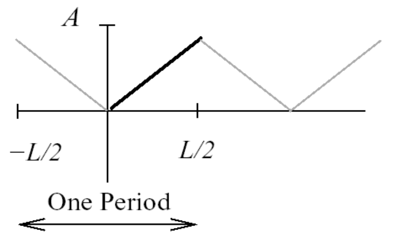


- **Recall…**
  - Corresponding Fourier expansion for an even function only contains (DC coefficient +) cosine terms

$$c_n = \frac{a_n - jb_n}{2} = \frac{1}{L}\int_0^L f(x)\, e^{-jn\omega x}\, dx$$

5.4.2 Half/Quarter-Range Series (cont.)

- **Major advantage of cosine-only series'**…
  - Only need to calculate real parts of complex coefficients c_n
    - i.e. imaginary parts = 0

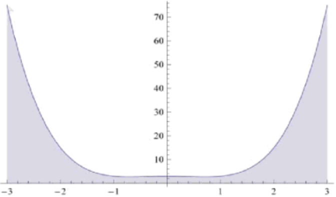
      (N.B. Always desirable to avoid Complex arithmetic from implementation point of view!)

- **Furthermore (Half-Range Series in general)…**
  - The integrals used to calculate the coeffs only need be evaluated over one half of the period
    - i.e. integral of an even function from –A to +A is simply twice the integral from 0 to +A
  - Desirable
    - Less computation involved!

$$a_0 = \frac{2}{L}\int_0^L f(x)\, dx = \frac{4}{L}\int_0^{L/2} f(x)\, dx,$$

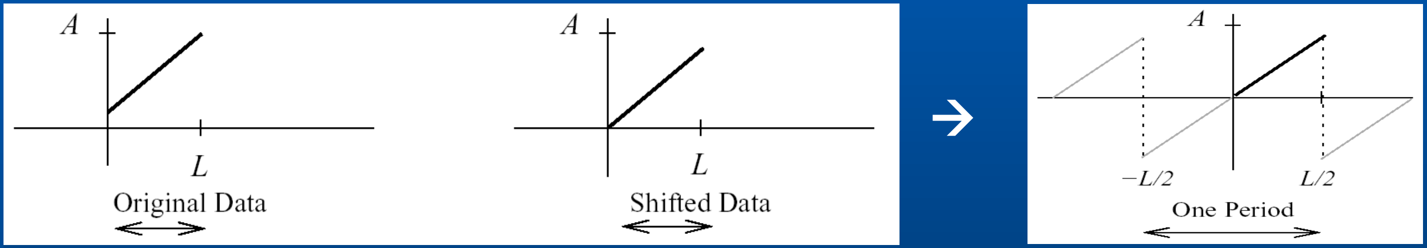
$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\omega x\, dx = \frac{4}{L}\int_0^{L/2} f(x)\cos n\omega x\, dx$$

$$b_n = 0$$



5.4.2 Half/Quarter-Range Series (cont.)

- **Alternatively (Half-Range Series)…**
  - Can use the selected data block to construct an *odd* periodic function
    - where *f(-x) = -f(x)*

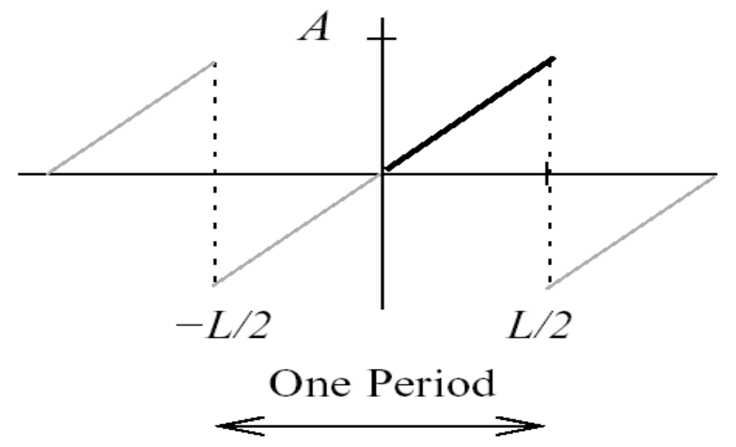


- **Recall…**
  - Corresponding Fourier expansion for an odd function only contains sine terms (+ no DC coefficient)

5.4.2 Half/Quarter-Range Series (cont.)

- **Disadvantage of sine series'…**
  - Usually introduces a large jump discontinuity at the join between two cycles (the constructed period)

    → slower convergence!

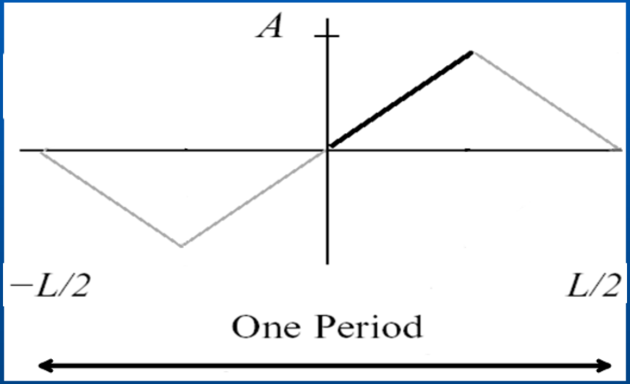


  - **Corollary of this**…
    - <u>Cosine series</u>' converge faster
    - Another way of putting this…
      - <u>Cosine series</u>' tend to have more signal energy concentrated in the lower frequency terms

        → More amenable to discarding higher frequency terms without signficant signal degredation! ☺

5.4.2 Half/Quarter-Range Series (cont.)

- **Quarter-Range Series?**
  - Could also synthesise periodic function where the selected data block forms one quarter of the constructed period



  - Advantages?
    - Quarter-Range Sine Series…
      - No jump discontinuities in the constructed periodic function
        → better convergence than half-range equivalent
    - Quarter-Range Cosine Series…
      - Expansion contains even harmonics only
        → No real advantage to this…
          → Takes as many even harmonics to represent the original function as it would with both even & odd terms

End of Chapter 5!